# QUADS

# Angular-Density Gravity from Discrete Angular Quanta

## Paper 4

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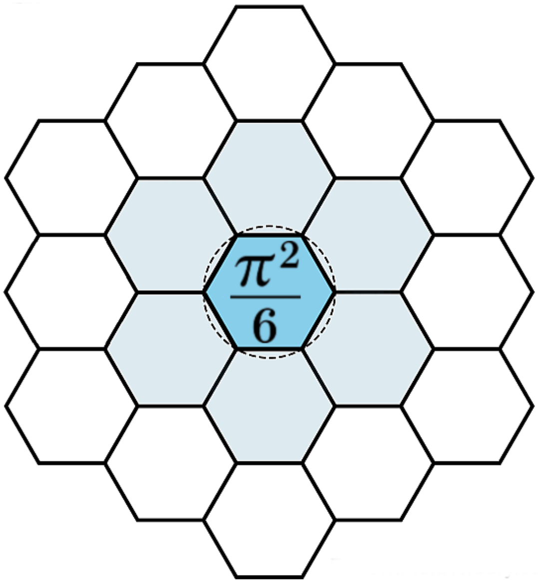


**Abstract**

Pattern Field Theory (PFT) models physical behaviour as transport, accumulation, and inward closure of discrete structural content on the Allen Orbital Lattice (AOL). QUADS IV derives a complete gravity sector from a single geometric primitive: the discrete angular quantum ("$\pi$-particle") interpreted as an irreducible angular update supported by a hexagonal substrate. Curvature is defined as angular deficit density, yielding a curvature tensor expressed directly in angular-density variables. An Einstein-like field equation is proposed in these variables, geodesic motion is derived as curvature-induced rotation of transport direction (without taking a metric as primitive), and weak-field observables—Newtonian acceleration, light bending, gravitational redshift/time dilation, gravitational waves, and perihelion precession—follow in a unified closure-and-transport framework.

## Overview and Scope

QUADS IV establishes the following chain:

**Q1**. **Angular quantum primitive:** the substrate supports discrete direction updates in units $\varepsilon_\theta$.

**Q2**. **Curvature from deficit:** curvature arises from angular deficit per minimal loop (plaquette).

**Q3**. **Angular-density curvature tensor:** curvature components are proportional to oriented angular-deficit densities.

**Q4**. **Gravity from closure:** inward closure responds to angular-density imbalance; gradients of the resulting potential drive acceleration.

**Q5**. **Dynamics:** angular-density disturbances propagate as waves with speed set by the lattice transport ceiling.

This paper is intended as a stable derivation layer: it defines variables, states admissibility conditions, and provides continuum-limit relations suitable for comparison with weak-field gravitational observables.

## Substrate and Kinematic Primitives

**Definition 1** (Hexagonal transport layer). *Let the lateral substrate be a regular hexagonal tiling with lattice spacing $a > 0$ and six nearest-neighbour directions. Motion across a lateral layer proceeds in straight steps of length $a$ between update events.*

**Definition 2** (Discrete angular quantum). *Let $\varepsilon_\theta > 0$ denote the minimal resolvable directional increment supported by the substrate. A directional update event at vertex $i$ changes the transport direction by*

$$\Delta\theta_i = N_i \varepsilon_\theta, \qquad N_i \in \mathbb{Z}.$$

*The $\pi$-particle is defined as a single irreducible angular update of magnitude $\varepsilon_\theta$.*

**Remark 1.** *If the substrate is hex-directional, a natural candidate is $\varepsilon_\theta = 2\pi/6 = \pi/3$, but QUADS IV keeps $\varepsilon_\theta$ as a parameter so that other admissible regimes can be expressed.*

## Discrete Curvature as Angular Deficit

**Definition 3** (Angular deficit on a minimal loop). *Consider a minimal closed loop (plaquette boundary) $\partial\Sigma$ in the lateral layer. Let $\sum_{\partial\Sigma} \Delta\theta$ be the sum of exterior turning angles along the loop. Define the deficit $\delta(\Sigma)$ by*

$$\sum_{\partial\Sigma} \Delta\theta = 2\pi - \delta(\Sigma).$$

*In discrete form, $\delta(\Sigma)$ is quantized:*

$$\delta(\Sigma) = M(\Sigma)\, \varepsilon_\theta, \qquad M(\Sigma) \in \mathbb{Z}.$$

**Definition 4** (Discrete Gaussian curvature density). *Let $A_\Sigma$ denote the area associated with plaquette $\Sigma$. Define*

$$K_\Sigma := \frac{\delta(\Sigma)}{A_\Sigma} = \frac{M(\Sigma)\,\varepsilon_\theta}{A_\Sigma}.$$

**Lemma 1** (Piecewise-linear curvature). *For a transport path composed of segments of length $a$, the local (discrete) curvature at vertex $i$ is*

$$\kappa_i = \frac{\Delta\theta_i}{a} = \frac{N_i\,\varepsilon_\theta}{a}.$$

## Angular-Density Curvature Tensor

**Definition 5** (Angular-deficit density field). *Let $\rho_\theta(x)$ denote the angular-quantum (deficit) density per unit area in the continuum limit:*

$$\rho_\theta(x) := \lim_{\Delta A \to 0} \frac{1}{\Delta A} \sum_{\Sigma \subset \Delta A} M(\Sigma).$$

*Then the continuum Gaussian curvature satisfies*

$$K(x) = \varepsilon_\theta\,\rho_\theta(x).$$

**Definition 6** (Oriented angular-density tensor). *In a 3D transport extension, angular deficits occur in oriented transport planes. Let $\rho_{ij}(x)$ be the deficit density associated with the $(i,j)$ transport plane. Define the curvature tensor proxy*

$$R_{ij}(x) := \varepsilon_\theta\,\rho_{ij}(x).$$

*Define its trace $R := \mathrm{tr}(R_{ij})$.*

**Definition 7** (Angular Einstein tensor). *Define the divergence-compatible combination*

$$G^{(\theta)}_{ij} := R_{ij} - \frac{1}{2} R\,\delta_{ij}.$$

**Remark 2.** *The role of $G^{(\theta)}_{ij}$ is to provide a geometric response tensor with the same structural purpose as the Einstein tensor: a symmetric combination of curvature data that is compatible with conservation constraints in the continuum limit.*

## Closure, Potential, and Newton Limit

**Definition 8** (Depth closure rate and time). *Let $\gamma(x) > 0$ denote the local inward closure (equilibration) rate. Local clock processes scale with closure progression:*

$$f(x) \propto \gamma(x).$$

**Proposition 1** (Potential proportional to angular density). *Assume that angular density acts as structural load that suppresses closure rate in weak field:*

$$\gamma(x) = \gamma_0\left(1 - \alpha\,\rho_\theta(x)\right),$$

*for small $\alpha\rho_\theta$. Define a gravitational potential proxy*

$$\Phi(x) := \beta\,\rho_\theta(x),$$

*with coupling $\beta > 0$. Then, after parameter identification, closure-rate modulation takes the form*

$$\frac{d\tau}{dt} = 1 - \frac{\Phi(x)}{c^2},$$

*where $c$ is the substrate transport ceiling and $\tau$ is local proper time.*

**Definition 9** (Acceleration as potential gradient)**.** *Define gravitational acceleration proxy in the lateral layer by*

$$g(x) := -\nabla\Phi(x).$$

**Lemma 2** (Poisson form in weak field)**.** *In a static regime where the effective source density is proportional to $\rho_\theta(x)$ and $\Phi \propto \rho_\theta$, the weak-field potential satisfies a Poisson-like equation*

$$\nabla^2\Phi(x) \propto \rho_\theta(x),$$

*recovering Newtonian scaling in the appropriate symmetry class.*

## Propagation Ceiling on a Hexagonal Lattice

**Definition 10** (Nearest-neighbour phase transport rule)**.** *Let $\theta(r,t)$ be a phase-like transport variable at lattice site $r$. Assume a nearest-neighbour hex-lattice wave rule*

$$\frac{d^2\theta(r,t)}{dt^2} = \Omega^2 \sum_{j=1}^{6} \left(\theta(r + ae_j, t) - \theta(r,t)\right),$$

*where $\{e_j\}_{j=1}^6$ are the six unit directions and $\Omega$ is a substrate coupling frequency.*

**Proposition 2** (Small-$k$ dispersion and transport ceiling)**.** *For plane-wave solutions $\theta(r,t) = \exp(i(k \cdot r - \omega t))$, the small-$ka$ dispersion is linear:*

$$\omega \approx \Omega a\sqrt{\frac{3}{2}}\,|k|.$$

*Hence the group velocity bound (transport ceiling) is*

$$c = \frac{d\omega}{d|k|} = \Omega a\sqrt{\frac{3}{2}}.$$

## Einstein-like Field Equation in Angular-Density Variables

**Definition 11** (Angular stress tensor)**.** *Let $T_{ij}^{(\theta)}$ denote the angular-transport stress tensor encoding directional transport loading associated with angular-density flow. A minimal symmetric form in a continuum regime is*

$$T_{ij}^{(\theta)} = \rho_\theta v_i v_j + P_\theta\,\delta_{ij},$$

*where $v$ is an effective transport velocity field and $P_\theta$ is an isotropic angular pressure.*

**Proposition 3** (Angular-density field equation)**.** *Postulate the field equation*

$$G_{ij}^{(\theta)} = \kappa_\theta\, T_{ij}^{(\theta)},$$

*with coupling constant $\kappa_\theta$ setting conversion between angular stress and curvature density.*

**Remark 3.** *This is "Einstein-like" in structure (curvature response to stress) but differs in primitives: curvature is built from discrete angular deficit density rather than a continuous metric as the fundamental entity.*

## Geodesic Transport Without Metric Primitivism

**Definition 12** (Curvature-induced rotation operator)**.** *Let* $\Omega^i{}_j(x)$ *be the local rotation generator for transport direction. In the minimal identification,*

$$\Omega^i{}_j(x) = \lambda\, R^i{}_j(x) = \lambda\, \varepsilon_\theta\, \rho^i{}_j(x),$$

*with normalization constant* $\lambda$.

**Proposition 4** (Angular-density geodesic equation)**.** *Let* $u^i = dx^i/ds$ *be the unit tangent of a path parameterized by arc-length* $s$. *Transport direction evolves by*

$$\frac{du^i}{ds} = -\Omega^i{}_j\, u^j.$$

*Equivalently, the trajectory satisfies*

$$\frac{d^2x^i}{ds^2} + \lambda\, \varepsilon_\theta\, \rho^i{}_j\, \frac{dx^j}{ds} = 0.$$

**Remark 4.** *In weak static regimes this reduces to Newton-like motion where acceleration is controlled by gradients of the potential associated with* $\rho_\theta$.

## Light Bending

For massless transport constrained to $|v| = c$, the direction changes while speed remains fixed. The net bending angle along a path $\Gamma$ is proportional to the integrated transverse angular density:

$$\Delta\theta \propto \int_\Gamma \varepsilon_\theta\, \rho_\perp(s)\, ds.$$

This reproduces the structural dependence of gravitational lensing on integrated curvature content along the ray path.

## Time Dilation and Gravitational Redshift

Assuming weak-field closure modulation

$$\frac{d\tau}{dt} = 1 - \frac{\Phi(x)}{c^2},$$

clock frequencies scale as $f(x) \propto d\tau/dt$. For emission at $e$ and observation at $o$:

$$\frac{f_o}{f_e} = \frac{1 - \Phi(o)/c^2}{1 - \Phi(e)/c^2} \approx 1 + \frac{\Phi(e) - \Phi(o)}{c^2},$$

yielding the standard weak-field gravitational redshift scaling.

## Gravitational Waves as Angular-Density Oscillations

**Definition 13** (Angular-density wave variables)**.** *Let* $\rho_{ij}(x,t) = \rho^{(0)}_{ij}(x) + \delta\rho_{ij}(x,t)$ *with* $\delta\rho_{ij}$ *a small perturbation.*

Assume conservative angular transport with inertia via an angular "momentum flux" field $P_{ij}$:

$$\partial_t \rho_{ij} + \nabla \cdot P_{ij} = 0, \qquad \partial_t P_{ij} = c^2 \nabla \rho_{ij}.$$

Then

$$\partial_t^2 \rho_{ij} - c^2 \nabla^2 \rho_{ij} = 0,$$

so angular-density curvature disturbances propagate as waves with speed $c$.

## Perihelion Precession Constraint

In weak-field orbital dynamics, the combined presence of (i) closure-based time dilation and (ii) curvature-induced direction rotation yields a first correction to Newton's effective potential with a characteristic $1/r^3$ term. In standard perturbative form this produces a perihelion advance per orbit:

$$\Delta\theta = \frac{6\pi GM}{a(1-e^2)c^2},$$

where $a$ is semi-major axis and $e$ is eccentricity. This matches the known weak-field general-relativistic structure when the coefficient is fixed by the same transport ceiling scale $c$ used throughout QUADS IV.

## Glossary

**AOL** Allen Orbital Lattice. The discrete hexagonal substrate used for lateral transport and update rules.

**Angular quantum $\varepsilon_\theta$**
Minimal resolvable direction increment supported by the substrate.

**$\pi$-particle** A discrete angular quantum event (an irreducible angular update) in the substrate.

**Angular deficit $\delta$**
Deviation of loop turning sum from $2\pi$; quantized as $\delta = M\varepsilon_\theta$.

**Angular density $\rho_\theta$**
Deficit-count density per unit area in a continuum limit.

**Curvature tensor proxy $R_{ij}$**
Defined by $R_{ij} = \varepsilon_\theta \rho_{ij}$ in oriented transport planes.

**Angular Einstein tensor $G_{ij}^{(\theta)}$**
$G_{ij}^{(\theta)} = R_{ij} - \frac{1}{2} R \delta_{ij}$.

**Closure rate $\gamma$**
Local inward equilibration rate; governs local clock progression in PFT.

**Transport ceiling $c$**
Maximum stable group velocity on the lattice (small-$k$ bound).

**Potential $\Phi$** Effective gravitational potential proxy proportional to $\rho_\theta$ in weak field.

## References


1. James Johan Sebastian Allen. *Pattern Field Theory (PFT) and the Allen Orbital Lattice (AOL): QUAD Derivations Working Set*. Internal derivation document family, 2026.

2. A. Einstein. *The Foundation of the General Theory of Relativity*. Annalen der Physik, 1916. (Conceptual comparator for field-equation structure.)

3. S. Weinberg. *Gravitation and Cosmology*. Wiley, 1972. (Weak-field limits and perihelion precession comparator.)

## Citation

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## Document Timestamp and Provenance

This document is part of Pattern Field Theory (PFT) and the Quantum Angular Density Substrate (QUADS) framework. It defines curvature as an angular-deficit density, derives an Einstein-like field equation in angular-density variables, derives geodesic transport without assuming a metric as primitive, and derives weak-field observables (Newton limit, light bending, gravitational redshift, wave propagation, perihelion precession) as consequences of closure and transport. 