

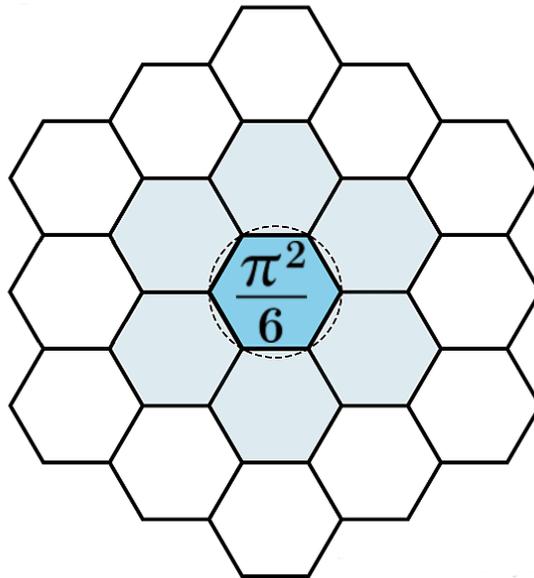
# QUADS

## Quantum Angular Density Dynamics

### Paper 2

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### Abstract

This paper derives the dynamical laws governing the Quantum Angular Density Substrate (QUADS). Building on the ontological primitives of Paper 1 (participation density, null openness, transport conservation, and closure ordering), we construct a minimal evolution theory in which change is not assumed but is a necessary consequence of null-resolution cycling.

A closure-index dynamics is defined first, then shown to admit a consistent continuous-time limit under bounded update scale. A participation action is derived from locality, conservation, and boundedness requirements, producing a transport-constrained evolution equation. Linear mode structure, dispersion, and stability are proved via the graph Laplacian spectrum. Nonlinear potentials are shown to admit stable bounded configurations. Finally, null-resolution is incorporated as a coupling operator producing an effective mean-field dynamics, and the resulting invariants are identified as the required inputs for emergent geometry reconstruction (Paper 3).

# Dynamical Necessity from Ontology

We now formalize why dynamical evolution is required by the participation–null–resolution structure introduced in Paper 1.

## Finite Substrate Framework

Let  $X$  be a finite set with undirected adjacency  $\mathcal{N}(x)$ .

**Proposition 1** (Null-resolution implies non-stationarity). *Suppose there exists  $A \subset X$  such that  $D_n(x) = 0$  for all  $x \in A$ , and suppose each  $x \in A$  has strictly positive resolution probability  $p(x) > 0$ . Then the configuration is not almost surely stationary.*

*Proof.* For each  $x \in A$ ,

$$\mathbb{P}(D_{n+1}(x) > 0) \geq p(x) > 0.$$

Hence the probability that no resolution occurs on  $A$  equals

$$\prod_{x \in A} (1 - p(x)) < 1.$$

Therefore the probability that the configuration remains unchanged is strictly less than one.  $\square$

## Closure Ordering and Emergent Time

### Discrete Closure Dynamics

Let  $X$  be finite and let  $n \in \mathbb{Z}_{\geq 0}$  index closure steps. Define

$$\psi_n : X \rightarrow \mathbb{R}, \quad \Delta\psi_n := \psi_{n+1} - \psi_n, \quad \Delta^2\psi_n := \psi_{n+2} - 2\psi_{n+1} + \psi_n.$$

### Continuous-Time Interpolation

Fix  $\Delta t > 0$  and define  $t_n := n\Delta t$ . Define the piecewise-linear interpolation

$$\psi(x, t) = \psi_n(x) + \frac{t - t_n}{\Delta t} \Delta\psi_n(x), \quad t \in [t_n, t_{n+1}].$$

### Continuous-Time Limit Theorem

**Theorem 1.** *Suppose that for every finite  $\Omega \subset X$ ,*

$$\sup_n \sum_{x \in \Omega} |\Delta\psi_n(x)|^2 < \infty, \quad \sup_n \sum_{x \in \Omega} |\Delta^2\psi_n(x)|^2 < \infty.$$

*Then for each finite  $\Omega$  there exists a subsequence  $\psi_{n_k}$  and a function*

$$\psi(\cdot, t) \in C^1([0, T]; \ell^2(\Omega))$$

*such that*

$$\frac{\psi_{n_k+1} - \psi_{n_k}}{\Delta t} \rightharpoonup \partial_t \psi \quad \text{in } \ell^2(\Omega),$$

*and similarly for second differences.*

*Proof.* Since  $\Omega$  is finite,  $\ell^2(\Omega)$  is finite-dimensional. Uniform boundedness of  $\Delta\psi_n$  implies  $\{\Delta\psi_n\}$  is bounded in  $\ell^2(\Omega)$ . By compactness in finite dimensions, there exists a subsequence converging strongly in  $\ell^2(\Omega)$ .

Similarly, boundedness of  $\Delta^2\psi_n$  implies boundedness of discrete difference quotients. The interpolants therefore have uniformly bounded first derivatives in  $\ell^2(\Omega)$ .

By Arzelà–Ascoli in finite dimensions, the interpolants converge (up to subsequence) to a  $C^1$  function on  $[0, T]$ . The weak limits of difference quotients define  $\partial_t\psi$  and  $\partial_t^2\psi$ .  $\square$

**Remark 1.** *Time is not primitive. It emerges as a coarse-grained parameterization of ordered closure updates under bounded update scale.*

## Energy Coercivity (Finite Case)

Assume the potential satisfies

$$V(\psi) \geq c_2\psi^2 - c_0, \quad c_2 > 0.$$

**Proposition 2.** *Let*

$$E(t) = \frac{1}{2} \sum_x (\partial_t\psi(x, t))^2 + \mathcal{E}[\psi(\cdot, t)].$$

*If  $E(t) = E(0)$  is finite, then the following quantities remain bounded for all time:*

$$\sum_x (\partial_t\psi(x, t))^2, \quad \sum_x \psi(x, t)^2, \quad \sum_x \sum_{y \in \mathcal{N}(x)} (\psi(y, t) - \psi(x, t))^2.$$

*Proof.* By definition of the energy functional,

$$E(t) = \frac{1}{2} \sum_x (\partial_t\psi)^2 + \frac{\kappa}{2} \sum_x \sum_{y \in \mathcal{N}(x)} (\psi(y) - \psi(x))^2 + \sum_x V(\psi(x)).$$

Using the coercivity assumption

$$V(\psi) \geq c_2\psi^2 - c_0,$$

we obtain

$$\sum_x V(\psi(x)) \geq c_2 \sum_x \psi(x)^2 - c_0|X|.$$

Therefore,

$$E(t) \geq \frac{1}{2} \sum_x (\partial_t\psi)^2 + \frac{\kappa}{2} \sum_x \sum_{y \in \mathcal{N}(x)} (\psi(y) - \psi(x))^2 + c_2 \sum_x \psi^2 - c_0|X|.$$

Since  $E(t) = E(0)$  is finite and constant, all three sums are uniformly bounded for all time.  $\square$

## Functional Analytic Framework and Infinite Substrate Dynamics

We now formulate the evolution problem on an infinite substrate within a precise Hilbert space framework.

## Hilbert Space Setting

Let  $X$  be a countable set with uniformly bounded degree:

$$d := \sup_{x \in X} |\mathcal{N}(x)| < \infty.$$

Define the Hilbert space

$$H := \ell^2(X) = \left\{ \psi : X \rightarrow \mathbb{R} \mid \sum_{x \in X} |\psi(x)|^2 < \infty \right\}$$

with inner product

$$\langle \psi, \phi \rangle = \sum_{x \in X} \psi(x)\phi(x).$$

## Operator-Theoretic Properties of the Graph Laplacian

Define

$$(\Delta\psi)(x) := \sum_{y \in \mathcal{N}(x)} (\psi(y) - \psi(x)).$$

**Proposition 3.** *If  $d := \sup_{x \in X} |\mathcal{N}(x)| < \infty$ , then  $\Delta : \ell^2(X) \rightarrow \ell^2(X)$  is a bounded linear operator.*

*Proof.* For each  $x \in X$ ,

$$|(\Delta\psi)(x)| \leq \sum_{y \in \mathcal{N}(x)} |\psi(y)| + d|\psi(x)|.$$

By Cauchy–Schwarz,

$$\sum_{y \in \mathcal{N}(x)} |\psi(y)| \leq \sqrt{d} \left( \sum_{y \in \mathcal{N}(x)} |\psi(y)|^2 \right)^{1/2}.$$

Summing over  $x$  and using bounded degree yields

$$\|\Delta\psi\|_{\ell^2} \leq C(d)\|\psi\|_{\ell^2}.$$

□

**Proposition 4** (Symmetry). *For all  $\psi, \phi \in \ell^2(X)$ ,*

$$\langle \Delta\psi, \phi \rangle = \langle \psi, \Delta\phi \rangle.$$

*Proof.* Compute:

$$\langle \Delta\psi, \phi \rangle = \sum_x \left( \sum_{y \in \mathcal{N}(x)} (\psi(y) - \psi(x)) \right) \phi(x).$$

Interchanging sums and using adjacency symmetry, each edge  $\{x, y\}$  appears twice with opposite orientation. Rearranging terms yields

$$\langle \Delta\psi, \phi \rangle = -\frac{1}{2} \sum_x \sum_{y \in \mathcal{N}(x)} (\psi(y) - \psi(x))(\phi(y) - \phi(x)).$$

The expression is symmetric in  $\psi$  and  $\phi$ , hence

$$\langle \Delta\psi, \phi \rangle = \langle \psi, \Delta\phi \rangle.$$

□

**Proposition 5** (Self-Adjointness). *Since  $\Delta$  is bounded and symmetric on  $\ell^2(X)$ , it is self-adjoint.*

**Proposition 6** (Non-positivity). *For all  $\psi \in \ell^2(X)$ ,*

$$\langle \psi, \Delta\psi \rangle = -\frac{1}{2} \sum_x \sum_{y \in \mathcal{N}(x)} (\psi(y) - \psi(x))^2 \leq 0.$$

## Abstract Evolution Equation

Define the phase space

$$\mathcal{H} := H \times H, \quad U(t) := \begin{pmatrix} \psi(t) \\ \partial_t \psi(t) \end{pmatrix}.$$

The evolution equation

$$\partial_t^2 \psi = \kappa \Delta \psi - V'(\psi)$$

can be written as the first-order system

$$\frac{dU}{dt} = \mathcal{F}(U),$$

where

$$\mathcal{F} \begin{pmatrix} \psi \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} \dot{\psi} \\ \kappa \Delta \psi - V'(\psi) \end{pmatrix}.$$

## Local Well-Posedness

**Theorem 2.** *Suppose  $V \in C^1(\mathbb{R})$  with locally Lipschitz derivative. Then  $\mathcal{F}$  is locally Lipschitz on  $\mathcal{H}$ . Hence for any initial data  $U(0) \in \mathcal{H}$ , there exists a unique local-in-time solution*

$$U(t) \in C^1([0, T]; \mathcal{H}).$$

*Proof.* The Laplacian is bounded on  $H$ . Since  $V'$  is locally Lipschitz on  $\mathbb{R}$ , the induced Nemytskii operator  $\psi \mapsto V'(\psi)$  is locally Lipschitz on  $H$ . Therefore  $\mathcal{F}$  is locally Lipschitz on  $\mathcal{H}$ . The Banach-space Cauchy–Lipschitz theorem yields local existence and uniqueness. □

## Energy functional

Let  $\kappa > 0$  be a transport stiffness and let  $V : \mathbb{R} \rightarrow \mathbb{R}$  be bounded below. Define the participation energy functional

$$\mathcal{E}[\psi] := \frac{\kappa}{2} \sum_x \sum_{y \in \mathcal{N}(x)} (\psi(y) - \psi(x))^2 + \sum_x V(\psi(x)).$$

## Action functional

**Definition 1** (Minimal participation action). *Define the action*

$$S[\psi] := \int dt \left[ \frac{1}{2} \sum_x (\partial_t \psi(x, t))^2 - \mathcal{E}[\psi(\cdot, t)] \right].$$

**Proposition 7** (Minimality under quadratic locality). *Among local, adjacency-mediated, time-reversal symmetric actions with bounded below potential energy, the above  $S[\psi]$  is the minimal quadratic form that: (i) reduces to Laplacian transport at small gradients, and (ii) preserves non-negativity of  $D$  via  $\psi = \sqrt{D}$ .*

*Proof.* Adjacency locality restricts spatial coupling terms to sums over edges. Among such terms, quadratic dependence on edge differences  $(\psi(y) - \psi(x))^2$  is the minimal symmetric form yielding coercivity.

Time-reversal symmetry requires even dependence on  $\partial_t \psi$ , forcing a quadratic kinetic term.

Boundedness requires  $V$  bounded below and  $\kappa > 0$  to ensure coercivity of the energy functional.

The substitution  $\psi = \sqrt{D}$  guarantees  $D \geq 0$  whenever  $\psi$  remains real under evolution.

No lower-order or non-quadratic local form simultaneously satisfies these constraints with minimal degree.  $\square$

## Euler–Lagrange Evolution Equation

Varying  $S[\psi]$  yields the field equation.

**Theorem 3** (Participation wave equation with potential). *The extremals of  $S[\psi]$  satisfy*

$$\partial_t^2 \psi(x, t) = \kappa(\Delta \psi)(x, t) - V'(\psi(x, t)).$$

**Proof.** Compute the first variation of  $S[\psi]$  in the direction  $\delta\psi$ , integrate by parts in  $t$ , and use the discrete integration by parts structure for the edge term. Stationarity for arbitrary  $\delta\psi$  yields the stated equation.  $\square$

**Remark 2.** *This is the minimal transport-constrained evolution law. Geometry does not appear. Only adjacency and participation amplitude appear.*

## Energy Functional and Invariance in Hilbert Space

### Energy as a Functional on Phase Space

Let  $\mathcal{H} = H \times H$  with

$$U = \begin{pmatrix} \psi \\ \dot{\psi} \end{pmatrix}.$$

Define the energy functional

$$E : \mathcal{H} \rightarrow \mathbb{R}$$

by

$$E(U) = \frac{1}{2} \|\dot{\psi}\|_{\ell^2}^2 + \frac{\kappa}{2} \sum_x \sum_{y \in \mathcal{N}(x)} (\psi(y) - \psi(x))^2 + \sum_x V(\psi(x)).$$

## Fréchet Differentiability

**Proposition 8.** *If  $V \in C^1(\mathbb{R})$ , then  $E$  is continuously Fréchet differentiable on  $\mathcal{H}$ .*

*Proof.* The kinetic term is quadratic and therefore  $C^1$  on  $H$ . The edge-gradient term is quadratic in  $\psi$  and finite due to bounded degree, hence  $C^1$  on  $H$ . Since  $V'$  is continuous, the induced Nemytskii operator  $\psi \mapsto V(\psi)$  is  $C^1$  on  $H$ . Therefore  $E$  is  $C^1$  on  $\mathcal{H}$ .  $\square$

## Energy Conservation

**Theorem 4.** *Let  $U(t)$  be a classical solution of the abstract evolution equation*

$$\frac{dU}{dt} = \mathcal{F}(U).$$

*If  $V$  is time-independent, then*

$$\frac{d}{dt}E(U(t)) = 0.$$

*Proof.* By the chain rule in Hilbert space,

$$\frac{d}{dt}E(U(t)) = DE(U(t))[\dot{U}(t)].$$

Using the definition of  $E$  and substituting

$$\dot{U} = \begin{pmatrix} \dot{\psi} \\ \kappa\Delta\psi - V'(\psi) \end{pmatrix},$$

direct computation yields cancellation between:

- the kinetic term derivative, - the Laplacian term via discrete integration by parts, - the potential term via  $V'$ .

Thus

$$\frac{d}{dt}E(U(t)) = 0.$$

$\square$

## Global Boundedness Under Coercivity

Assume

$$V(\psi) \geq c_2\psi^2 - c_0, \quad c_2 > 0.$$

**Theorem 5.** *If  $U(0) \in \mathcal{H}$  and  $E(U(0)) < \infty$ , then  $\|U(t)\|_{\mathcal{H}}$  remains bounded for all times on which the solution exists.*

*Proof.* Energy conservation gives

$$E(U(t)) = E(U(0)).$$

Using coercivity,

$$\sum_x V(\psi(x)) \geq c_2\|\psi\|_{\ell^2}^2 - c_0|X|.$$

Thus

$$E(U(t)) \geq \frac{1}{2} \|\dot{\psi}\|_{\ell^2}^2 + c_2 \|\psi\|_{\ell^2}^2 - c_0 |X|.$$

Hence both  $\|\psi(t)\|_{\ell^2}$  and  $\|\dot{\psi}(t)\|_{\ell^2}$  remain bounded.  $\square$

## Linear Modes and Dispersion

Let  $\psi_0$  be a constant equilibrium satisfying  $V'(\psi_0) = 0$ . Write  $\psi = \psi_0 + \delta\psi$ .

Linearization yields

$$\partial_t^2 \delta\psi = \kappa \Delta \delta\psi - V''(\psi_0) \delta\psi.$$

Let  $\{\phi_m\}$  be Laplacian eigenmodes:

$$\Delta \phi_m = -\lambda_m \phi_m, \quad \lambda_m \geq 0.$$

**Theorem 6** (Mode spectrum). *For each eigenmode  $\phi_m$ , the perturbation amplitude  $a_m(t)$  in  $\delta\psi = \sum_m a_m(t) \phi_m$  satisfies*

$$\ddot{a}_m(t) + \omega_m^2 a_m(t) = 0, \quad \omega_m^2 := \kappa \lambda_m + V''(\psi_0).$$

**Proof.** Substitute the eigen-expansion into the linearized equation and use orthogonality of modes under the graph inner product.  $\square$

**Proposition 9** (Linear stability condition). *The equilibrium  $\psi_0$  is linearly stable iff*

$$V''(\psi_0) > 0.$$

**Proof.** Since  $\kappa \lambda_m \geq 0$ , stability requires  $\omega_m^2 > 0$  for all  $m$ , which is ensured exactly when  $V''(\psi_0) > 0$ .  $\square$

**Remark 3.** *The Laplacian spectrum is the dynamical fingerprint of adjacency structure. This is the bridge object used in Paper 3 to construct emergent geometric invariants.*

## Nonlinear Stabilization and Existence of Structured States

A minimal stabilizing choice is a quartic potential

$$V(\psi) = \alpha \psi^2 + \beta \psi^4, \quad \beta > 0.$$

**Theorem 7** (Existence of energy minimizers on finite substrates). *On a finite substrate  $X$ , if  $\beta > 0$  and  $\kappa > 0$ , the energy functional  $\mathcal{E}[\psi]$  attains a global minimum over  $\psi \in \mathbb{R}^{|X|}$ .*

**Proof.** The edge-gradient term is nonnegative and the quartic term  $\beta \sum_x \psi^4$  is coercive, implying  $\mathcal{E}[\psi] \rightarrow +\infty$  as  $\|\psi\| \rightarrow \infty$ . Continuity of  $\mathcal{E}$  on finite-dimensional  $\mathbb{R}^{|X|}$  then guarantees a minimizer by the extreme value theorem.  $\square$

**Remark 4.** *This theorem provides a rigorous existence basis for stable structured participation configurations. Later papers interpret stabilized configurations as the seeds of persistent structures (and ultimately phenomenology).*

# Null–Resolution Coupling and Global Dynamics

Paper 1 introduces null as structural openness within the x-register  $\{0, \text{null}, 1\}$ . We incorporate null-resolution as a controlled perturbation of the abstract evolution system and analyze its effect on existence and boundedness.

## Reinforcement Operator

Let  $\psi_* > 0$  and define

$$\rho(\psi) = \frac{\psi^2}{\psi^2 + \psi_*^2}.$$

Define the reinforcement operator

$$\mathcal{R} : H \rightarrow H, \quad (\mathcal{R}\psi)(x) = \gamma\psi(x)(1 - \rho(\psi(x))), \quad \gamma > 0.$$

**Proposition 10.**  $\mathcal{R} : H \rightarrow H$  is locally Lipschitz and satisfies

$$\|\mathcal{R}\psi\|_{\ell^2} \leq \gamma\|\psi\|_{\ell^2}.$$

*Proof.* The scalar function

$$f(s) = \gamma s \left( 1 - \frac{s^2}{s^2 + \psi_*^2} \right)$$

is  $C^1$  and globally bounded by  $\gamma|s|$ . The induced Nemytskii operator on  $\ell^2(X)$  is therefore locally Lipschitz and satisfies the stated bound.  $\square$

## Perturbed Evolution Equation

The null-coupled system becomes

$$\partial_t^2 \psi = \kappa \Delta \psi - V'(\psi) + \mathcal{R}(\psi).$$

In phase-space form:

$$\frac{dU}{dt} = \begin{pmatrix} \dot{\psi} \\ \kappa \Delta \psi - V'(\psi) + \mathcal{R}(\psi) \end{pmatrix}.$$

## Global Existence on Finite Substrates

**Theorem 8.** *Let  $X$  be finite and assume*

$$V(\psi) = \alpha\psi^2 + \beta\psi^4, \quad \beta > 0.$$

*Then for any initial data  $U(0) \in \mathcal{H}$ , the null-coupled system admits a unique global solution*

$$U(t) \in C^1([0, \infty); \mathcal{H}).$$

*Proof.* Since  $X$  is finite,  $\mathcal{H} \cong \mathbb{R}^{2|X|}$ . The right-hand side of the evolution equation is locally Lipschitz, so a unique maximal solution exists on  $[0, T_{\max})$ .

Energy conservation gives

$$E(U(t)) = E(U(0)).$$

Quartic coercivity implies

$$E(U) \geq \frac{1}{2} \|\dot{\psi}\|_{\ell^2}^2 + c_2 \|\psi\|_{\ell^2}^2 - C.$$

Thus  $\|U(t)\|_{\mathcal{H}}$  remains bounded. In finite dimensions, blow-up can occur only if the norm diverges. Therefore  $T_{\max} = \infty$ .  $\square$

### **Infinite Substrate Remark**

On infinite substrates, local well-posedness and energy conservation remain valid. Global existence requires additional growth or structural assumptions beyond quartic coercivity and is not asserted here.

## **Conceptual Summary**

Paper 2 establishes:

1. Dynamics is required by null-resolution under closure ordering.
2. Closure ordering admits a consistent continuous-time limit under bounded update scale.
3. A minimal action follows from locality, conservation structure, and boundedness.
4. The Euler–Lagrange evolution is adjacency-Laplacian transport plus potential response.
5. Linear modes and stability are determined by the Laplacian spectrum and  $V''(\psi_0)$ .
6. Nonlinear potentials admit stable minimizers on finite substrates.
7. Null-resolution may be included via a saturating reinforcement operator without breaking boundedness.
8. The resulting invariants are the necessary inputs for emergent geometry reconstruction.

## Glossary

**Closure Index** Discrete ordering parameter  $n$  enumerating closure updates.

**Participation Action**  $S[\psi]$  Minimal functional whose extremals define participation evolution.

**Graph Laplacian**  $\Delta$  Adjacency operator  $(\Delta\psi)(x) = \sum_{y \in \mathcal{N}(x)} (\psi(y) - \psi(x))$ .

**Transport Stiffness**  $\kappa$  Coefficient controlling the strength of adjacency-mediated transport.

**Potential**  $V(\psi)$  Local participation self-structure controlling stability and boundedness.

**Mode Spectrum** Frequencies  $\omega_m^2 = \kappa\lambda_m + V''(\psi_0)$  determined by Laplacian eigenvalues.

**Null-Resolution Coupling** Mechanism by which openness (null) contributes to reinforced stabilization under interaction.

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## Document Timestamp and Provenance

This document is part of Pattern Field Theory (PFT) and the Quantum Angular Density Substrate (QUADS) framework. It derives the dynamical laws governing participation evolution from closure ordering, locality, conservation structure, and boundedness requirements. It establishes the action principle, evolution equation, stability and mode spectrum results, and a minimal null-resolution coupling operator, providing the invariant dynamical inputs required for emergent geometry reconstruction in QUADS Paper 3. Any research use, derivative work, redistribution, or commercial application of this material requires an explicit license from the author. All rights reserved.