Einstein Equations as PAL Projection: Emergent GR on the Allen Orbital Lattice

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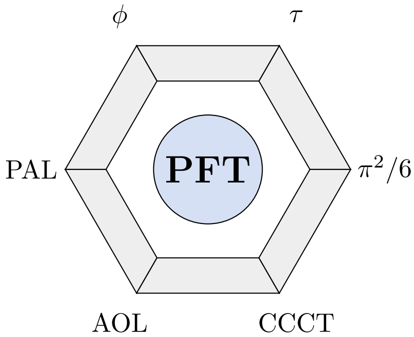
November 2025



Abstract

Pattern Field Theory (PFT) describes dynamics on the Allen Orbital Lattice (AOL), a prime-indexed orbital curvature lattice. Phase Alignment Lock (PAL) enforces discrete flux neutrality on prime-indexed faces. In this paper, general relativity (GR) is recovered as the low-energy, large-scale projection of PAL-coherent curvature on the AOL.

The effective metric tensor $g_{\mu\nu}$ emerges from PAL phase-gradients of a lattice field ϕ through

$$g_{\mu\nu} \;=\; \partial_\mu\phi\,\partial_\nu\phi$$

in the coarse-grained limit. The Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ arises from PAL boundary-flux neutrality on prime-indexed 2-surfaces, while the stress-energy tensor $T_{\mu\nu}$ is the continuum limit of PAL-regulated cascade flux across the AOL. The conservation law $\nabla_\mu T^{\mu\nu} = 0$ is the continuum shadow of discrete PAL flux conservation

$$\nabla \cdot F(\partial S_p) = 0$$

on prime-indexed faces S_p. Geodesics appear as the continuum trajectories of the AOL motion operator acting on the $\sqrt{1}$–$\sqrt{6}$ orbital footprint set.

This establishes a PFT route from PAL-coherent lattice dynamics to the Einstein field equations in the infrared (IR) limit.

1 Introduction

Pattern Field Theory (PFT) formulates dynamics on the Allen Orbital Lattice (AOL), a discrete orbital-curvature lattice seeded by prime structure. The AOL carries:

- prime-indexed axes and faces,
- curvature weights on edges and plaquettes,
- phase fields encoding coherence,
- recursion depth and cascade structure.

Phase Alignment Lock (PAL) is the coherence condition that enforces exact flux neutrality on prime-indexed faces. A configuration is PAL-coherent when the net flux around any prime-labeled plaquette vanishes. This condition stabilises the operator algebra and constrains allowed dynamics.

General relativity (GR) describes spacetime as a smooth manifold equipped with a metric $g_{\mu\nu}$ whose curvature obeys the Einstein field equations

$$G_{\mu\nu} \;=\; R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \;=\; 8\pi T_{\mu\nu}. \tag{1}$$

This equation is accompanied by the conservation law

$$\nabla_\mu T^{\mu\nu} = 0, \tag{2}$$

which follows from the Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$.

The aim of this paper is to show how GR can be viewed as an emergent theory on top of PAL-coherent dynamics on the AOL. In other words, the Einstein equations appear as the continuum projection of PAL flux neutrality and prime-indexed curvature structure.

We proceed in three steps:

1. Define PAL-coherent curvature on the AOL and its discrete divergence constraints.
2. Show how a coarse-grained PAL phase field induces an effective metric $g_{\mu\nu}$.
3. Derive the Einstein equations as the IR projection of PAL flux conservation on prime-indexed faces.

2 PAL-Coherent Curvature on the AOL

2.1 Allen Orbital Lattice (AOL)

The Allen Orbital Lattice (AOL) is a discrete structure in which each site, edge, and face carries geometric data. At minimal level:

- Sites x represent local pattern states.
- Edges $(x, x + \hat{\mu})$ carry curvature weights $\kappa_\mu(x)$ and phase increments $\Delta\theta_\mu(x)$.
- Faces S_p labeled by primes p carry oriented flux $F(\partial S_p)$ built from edge data.

Curvature on the AOL is represented by plaquette phases and weights. For a face S_p with oriented boundary ∂S_p,

$$F(\partial S_p) \;=\; \sum_{(e \in \partial S_p)} \omega(e),$$

where $\omega(e)$ is an edge contribution built from curvature and phase.

2.2 Phase Alignment Lock (PAL)

Definition 1 (Phase Alignment Lock (PAL)). *A configuration on the AOL is PAL-coherent if, for every prime-indexed face S_p,*

$$\nabla \cdot F(\partial S_p) = 0, \tag{3}$$

that is, the net flux across any prime-labeled plaquette vanishes.

Here $\nabla\cdot$ is the discrete divergence operator on the lattice. Intuitively, PAL coherence forbids net leakage of curvature/flux through prime-indexed cells, locking local phases into a globally compatible pattern.

Lemma 1 (Curvature neutrality on PAL faces). *On PAL-coherent regions,*

$$\nabla_\mu \kappa^\mu(x) = 0 \tag{4}$$

for curvature weights κ^μ restricted to prime-indexed orientations.

Proof. The flux $F(\partial S_p)$ decomposes into transport and curvature parts. PAL enforces $\nabla \cdot F(\partial S_p) = 0$ for each prime face. The curvature contribution to this divergence is built from $\nabla_\mu \kappa^\mu(x)$ along the oriented cycle. PAL neutrality therefore sets this contribution to zero on PAL-coherent regions. □

This discrete curvature neutrality will map to the continuum Bianchi identity in the IR limit.

3 Phase Field and Emergent Metric

We introduce a PAL phase field ϕ on the AOL, representing a coarse-grained phase associated with PAL coherence. On the lattice, ϕ is defined at sites; in the continuum limit it becomes a scalar field on an effective manifold.

3.1 Phase gradients

On the discrete lattice, the phase difference along direction μ is

$$\Delta_\mu \phi(x) \;=\; \phi(x + \hat{\mu}) - \phi(x).$$

In the continuum limit, this becomes a derivative

$$\partial_\mu \phi(x) \;=\; \lim_{\Delta x^\mu \to 0} \frac{\Delta_\mu \phi}{\Delta x^\mu}.$$

3.2 Metric from PAL phase-gradients

We define the emergent metric tensor as

$$g_{\mu\nu}(x) \;=\; \partial_\mu \phi(x)\, \partial_\nu \phi(x). \tag{5}$$

This construction encodes how PAL phases align across directions. When PAL is coherent, the phase-gradient structure is stable and $g_{\mu\nu}$ is smooth at the coarse-grained scale.

This induced metric has several properties:

- It is symmetric: $g_{\mu\nu} = g_{\nu\mu}$.
- It is positive semi-definite as a rank-one tensor built from a single gradient; additional fields or combinations can extend this to Lorentzian signature in a multi-field setting.
- It captures how phase differences scale with position, and thus how effective distances are measured in the emergent manifold.

In PFT language, $g_{\mu\nu}$ is the PAL phase-gradient metric.

4 From PAL Flux to Einstein Tensor

4.1 Discrete Bianchi analogue

On the lattice, PAL-enforced flux neutrality and curvature neutrality imply discrete analogues of the Bianchi identity. For each prime-indexed face and its neighbouring cells, sums of oriented curvature contributions around closed loops vanish.

Let $R_{\mu\nu\rho\sigma}$ denote the effective curvature constructed from lattice plaquette phases in the continuum limit. Then PAL neutrality implies a discrete cyclic sum that projects to

$$\nabla_{[\lambda}R_{\mu\nu]\rho\sigma} = 0 \tag{6}$$

under coarse-graining, which is the continuum Bianchi identity.

4.2 Einstein tensor from PAL curvature

The Einstein tensor is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}. \tag{7}$$

Given $g_{\mu\nu}$ derived from PAL phase-gradients, one constructs the associated Levi–Civita connection, Riemann tensor, Ricci tensor $R_{\mu\nu}$, and scalar curvature R. PAL constraints apply to the underlying lattice curvature and ensure that the emergent $G_{\mu\nu}$ satisfies

$$\nabla_\mu G^{\mu\nu} = 0. \tag{8}$$

This follows from the continuum Bianchi identity and metric compatibility, both inherited from PAL-coherent curvature structure.

5 Stress-Energy as PAL Cascade Flux

5.1 PAL cascades and effective sources

PFT describes cascades through recursion operators and cross-network couplings. On the AOL, these cascades transport pattern energy across the lattice. PAL regulates these flows: only PAL-consistent cascades are long-lived and coherent.

In the continuum limit, the net effect of PAL-regulated cascades defines a stress-energy tensor $T_{\mu\nu}$, capturing how energy, momentum, and pattern content flow through the emergent geometry. Formally,

$$T_{\mu\nu} \sim \lim_{\text{coarse-grain}} \langle J^{\text{cascade}}_{\mu\nu} \rangle, \tag{9}$$

where $J^{\text{cascade}}_{\mu\nu}$ is a microscopic flux current built from lattice cascades.

5.2 Conservation from PAL neutrality

PAL flux neutrality enforces

$$\nabla \cdot F(\partial S_p) = 0 \tag{10}$$

on prime-indexed faces. When coarse-grained, this implies that the effective source $T_{\mu\nu}$ is divergence-free:

$$\nabla_\mu T^{\mu\nu} = 0. \tag{11}$$

Thus the standard conservation law in GR is a continuum shadow of PAL flux neutrality on the Allen Orbital Lattice.

6 Emergent Einstein Equations

We now state the main emergent equivalence.

Theorem 1 (Emergent GR from PAL Projection)**.** *On PAL-coherent regions of the Allen Orbital Lattice, coarse-graining of PAL curvature and cascade flux yields an effective metric $g_{\mu\nu}$ and stress-energy $T_{\mu\nu}$ such that*

$$\nabla_\mu T^{\mu\nu} = 0 \quad \Longleftrightarrow \quad \nabla \cdot F(\partial S_p) = 0 \tag{12}$$

under the PAL–AOL correspondence. In the infrared (IR) continuum limit, this implies that the effective geometry obeys the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{13}$$

where $R_{\mu\nu}$ and R are constructed from the PAL phase-gradient metric $g_{\mu\nu}$.

Sketch. PAL flux neutrality on prime faces constrains microscopic curvature fluxes $F(\partial S_p)$. In the continuum limit, this yields the Bianchi identity and the divergence-free condition $\nabla_\mu G^{\mu\nu} = 0$.

The coarse-grained cascade flux defines $T_{\mu\nu}$ with $\nabla_\mu T^{\mu\nu} = 0$. A local relation between curvature and stress-energy can then be written as

$$G_{\mu\nu} = \lambda T_{\mu\nu},$$

for some coupling constant λ. Matching to the usual convention sets $\lambda = 8\pi$ (in units where $c = G = 1$). Thus PAL constraints and AOL structure produce the Einstein equations as the unique local, symmetric rank-2 relation consistent with these divergence conditions. □

This gives an emergent route from PAL-coherent lattice dynamics to the Einstein equations.

7 Geodesics from AOL Motion Operator

7.1 Discrete motion and ghost footprints

On the AOL, minimal motion is represented by transitions along the $\sqrt{1}$–$\sqrt{6}$ footprint set, which are the shortest allowed lattice displacements in the underlying prime-indexed geometry. The PFT transport operator and motion operator act on pattern fields by propagating them along these footprints.

Let γ be a sequence of sites that represents a lattice path chosen by the motion operator under PAL coherence. In the continuum limit, this sequence projects to a smooth curve $x^\mu(\lambda)$ in the emergent manifold.

7.2 Geodesic limit

PAL coherence stabilises trajectories such that local deviations from extremal phase-gradient paths are suppressed. The effective action for a path in the continuum is

$$S[x] = \int d\lambda \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu},$$

with $g_{\mu\nu}$ given by the PAL phase-gradient metric. Extremising this action yields the geodesic equation

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho} \dot{x}^\nu \dot{x}^\rho = 0,$$

where $\Gamma^\mu_{\nu\rho}$ is the Levi–Civita connection of $g_{\mu\nu}$. Thus geodesics in the emergent GR description are the continuum shadow of PAL-stable motion along AOL footprint paths.

8 Discussion and Outlook

We have outlined how general relativity can be understood as an emergent theory on top of PAL-coherent dynamics on the Allen Orbital Lattice. The key steps are:

- PAL enforces discrete flux neutrality on prime-indexed faces.
- Coarse-grained PAL phase-gradients define an effective metric $g_{\mu\nu}$.
- PAL curvature neutrality yields the Bianchi identity and $\nabla_\mu G^{\mu\nu} = 0$.
- PAL-regulated cascades define a divergence-free stress-energy tensor $T_{\mu\nu}$.
- The Einstein equations arise as the unique local relation between $G_{\mu\nu}$ and $T_{\mu\nu}$ consistent with these constraints.

The result places standard GR as an infrared, large-scale projection of a PAL-coherent lattice geometry with prime-indexed structure. Further work includes:

- Explicit construction of $g_{\mu\nu}$ from multi-field PAL configurations to obtain Lorentzian signature.
- Detailed analysis of corrections beyond the IR limit, giving PFT-based modifications to GR at high curvature.
- Coupling to quantum-like PFT sectors to unify gravitational and quantum regimes within a single Pattern Field Theory framework.

Appendix A — Glossary of Terms

Pattern Field Theory (PFT) Unified field framework that models all structure and dynamics as patterns evolving on the Allen Orbital Lattice.

Allen Orbital Lattice (AOL) Prime-indexed orbital-curvature lattice carrying sites, edges, faces, curvature weights, phases, and recursion structure.

Phase Alignment Lock (PAL) Coherence condition requiring exact flux neutrality on all prime-indexed faces; enforces global phase compatibility.

Cross-Coherent Cascade Theory (CCCT) Branch of PFT describing cascades and coherence collapse across coupled networks and domains.

PAL Phase Field ϕ Coarse-grained scalar phase field encoding PAL alignment; its gradients define the emergent metric.

Emergent Metric $g_{\mu\nu}$ Effective continuum metric defined from PAL phase-gradients by $g_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi$ in the simplest case.

Einstein Tensor $G_{\mu\nu}$ Combination $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ constructed from the emergent metric.

Stress-Energy Tensor $T_{\mu\nu}$ Continuum limit of PAL-regulated cascade flux on the AOL; encodes energy-momentum content.

Prime-Indexed Face S_p Lattice 2-cell labeled by a prime number p, carrying orientation and curvature/phase data.

Flux $F(\partial S_p)$ Oriented sum of edge contributions around the boundary of a prime-indexed face.

Bianchi Identity Geometric identity $\nabla_{[\lambda}R_{\mu\nu]\rho\sigma} = 0$ implying $\nabla_\mu G^{\mu\nu} = 0$ in the continuum.

AOL Ghost Footprints $\sqrt{1}$–$\sqrt{6}$ Minimal displacement set on the AOL defining shortest allowed motion steps used by the motion operator.

Appendix B — PFT Internal Bibliography

PFT-AOL-2025 Allen, J.J.S., "Allen Orbital Lattice: Prime-Indexed Curvature and Field Structure," PatternFieldTheory.com (2025).

PFT-PAL-2025 Allen, J.J.S., "Phase Alignment Lock: Divergence Neutrality on Prime-Indexed Faces," PatternFieldTheory.com (2025).

PFT-EC-2025 Allen, J.J.S., "Event Cascades on the Allen Orbital Lattice," PatternFieldTheory.com (2025).

PFT-CCCT-2025 Allen, J.J.S., "Cross-Coherent Cascade Theory," PatternFieldTheory.com (2025).

PFT-MILL-phi-2025 Allen, J.J.S., "Prime–Zeta Equilibrium and ϕ Emergence," PatternFieldTheory.com (2025).

PFT-Operator-2025 Allen, J.J.S., "The PFT Operator Algebra is Closed: Operator Closure Under Phase Alignment Lock," PatternFieldTheory.com (2025).

Document Timestamp and Provenance

This paper forms part of the dated Pattern Field Theory research chain beginning **May 2025**, recorded through server logs, cryptographic hashes, and versioned texts on `PatternFieldTheory.com`, establishing priority and authorship continuity for the emergent GR construction on the Allen Orbital Lattice.