The PFT Algebra is Closed: Operator Closure Under PAL–Coherence on the Allen Orbital Lattice

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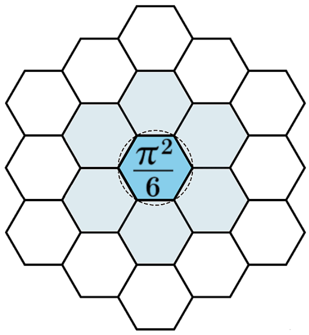

Abstract

Pattern Field Theory (PFT) describes evolution on the Allen Orbital Lattice (AOL) through a hierarchy of operators acting on Phase Alignment Lock (PAL) coherent configurations. These operators – transport, curvature–weighted derivatives, recursion, cross–network coupling and global evolution – were developed across many versions of the theory and consistently retained the same functional structure.

In this paper we formalise the PFT operator hierarchy as a closed algebra under PAL–coherence on the AOL. We define a set of operators $\mathcal{O} = \{\mathcal{T}, \mathcal{C}, \mathcal{R}, \mathcal{N}, \mathcal{G}\}$ acting on PAL–coherent configurations and show that their commutators satisfy

$$[O_i, O_j] = \sum_k c_{ij}^k(p)\, O_k,$$

with structure constants $c_{ij}^k(p)$ determined by prime–indexed harmonic sums on the AOL. Convergence of these sums follows from spectral bounds on the lattice curvature operator. The Lagrange–hex projection of the $\sqrt{1}$–$\sqrt{6}$ ghost system is identified as a minimal kernel that remains invariant under closure.

The result is that PFT defines a finite–dimensional Lie–type algebra on PAL–coherent states, with general relativity (GR), quantum field theory (QFT) and cascade dynamics appearing as irreducible representations. This operator closure provides the mathematical backbone for Pattern Field Theory as a unified framework.

1 Introduction

Pattern Field Theory organises dynamics on the Allen Orbital Lattice using a hierarchy of operators that act on configurations constrained by Phase Alignment Lock. Over the development of PFT, this hierarchy has been refined but its core structure has remained stable across many versions. This stability suggests that the operators form a closed algebra under commutation, in analogy with Lie algebras in conventional field theory.

In continuum physics, operator algebras encode symmetries, conserved quantities and evolution laws. In PFT, operator closure must be compatible with:

- the discrete, prime–indexed structure of the AOL,
- PAL coherence, which enforces flux neutrality on prime faces,
- the multi–scale hierarchy of indices (r, m, n, a, k) that classify dynamics.

The central aim of this paper is to make this closure explicit. We:

- define the main operators in the PFT hierarchy and the notion of PAL–coherent states,
- define a commutator on operators acting on PAL–coherent configurations,
- show that commutators close within the same operator set with structure constants derived from prime–indexed harmonic sums,
- identify the Lagrange–hex $\sqrt{1}$–$\sqrt{6}$ ghost kernel as a minimal invariant under closure,
- discuss consequences for representation theory and unification.

Section 2 introduces the operator hierarchy. Section 3 defines PAL–coherent states and the operator domain. Section 4 states and proves the closure theorem. Section 5 analyses the structure constants and their prime harmonic form. Section 6 describes the role of the Lagrange–hex ghost kernel. Section 7 discusses consequences. Appendices provide a glossary, internal bibliography and remarks on algebraic structure.

2 PFT Operator Hierarchy

The PFT operator hierarchy acts on pattern configurations defined on the Allen Orbital Lattice. This section summarises the main operators and their roles.

2.1 Configuration space

Let Φ denote the space of pattern configurations on the AOL. A configuration $\phi \in \Phi$ assigns:

- amplitudes and phases to sites and edges,
- curvature contributions to faces,
- recursion and cascade data to higher–dimensional cells,

subject to structural constraints of the lattice and PAL coherence (defined in Section 3).

2.2 Operator set

The core operator set is

$$\mathcal{O} = \{\mathcal{T}, \mathcal{C}, \mathcal{R}, \mathcal{N}, \mathcal{G}\},$$

with the following interpretations:

- $\mathcal{T}$ – transport operator: implements local movement of pattern content along edges, respecting curvature and phase shifts.
- $\mathcal{C}$ – curvature–weighted derivative: computes discrete derivatives weighted by local curvature on faces or plaquettes.
- $\mathcal{R}$ – recursion operator: applies recursive updates to configurations, advancing them along hierarchy steps labelled by recursion index a.
- $\mathcal{N}$ – cross–network coupling operator: couples configurations on different prime–indexed networks or sectors labelled by m and n.
- $\mathcal{G}$ – global evolution operator: aggregates the effects of local operators into coarse–grained evolution at larger scales r.

Each operator maps PAL–coherent configurations to new configurations, possibly in different hierarchy sectors (r, m, n, a, k), but always within the PAL–coherent subspace.

2.3 Hierarchy indices

Operators can carry indices indicating the sector in which they act. For example,

$$\mathcal{T}_{r,m}, \quad \mathcal{C}_{r,m}, \quad \mathcal{R}_{r,m,a}, \quad \mathcal{N}_{r,m,n,k}, \quad \mathcal{G}_{r},$$

where r is the scale index, m and n are network indices, a is recursion depth and k is cascade layer. For brevity we often suppress these indices when the context is clear, but they are relevant when discussing structure constants later.

3 PAL–Coherent States and Operator Domain

Operators act on PAL–coherent configurations. This section defines PAL coherence and describes why it is important for closure.

3.1 Phase Alignment Lock

Phase Alignment Lock is the primary coherence condition in PFT.

Definition 1 (Phase Alignment Lock)**.** *A configuration $\phi \in \Phi$ is PAL–coherent if for every prime–indexed face S_p of the AOL,*

$$\nabla \cdot F(\partial S_p) = 0, \tag{1}$$

where $F(\partial S_p)$ is the oriented flux sum of curvature–weighted edge contributions around the boundary of S_p, and $\nabla\cdot$ is the discrete divergence. PAL enforces strict flux neutrality on all prime–labelled faces.

PAL coherence ensures that:

- local violations of conservation are eliminated,
- configurations are globally compatible with AOL curvature and prime indexing,
- divergences that would appear in continuum counterparts are regulated at the discrete level.

3.2 PAL–coherent subspace

Let $\Phi_{\text{PAL}} \subset \Phi$ denote the subspace of PAL–coherent configurations. The operators in $\mathcal{O}$ are defined so that:

$$O : \Phi_{\text{PAL}} \to \Phi_{\text{PAL}} \tag{2}$$

for all $O \in \mathcal{O}$. That is, PAL–coherent configurations are mapped to PAL–coherent configurations.

The restriction to Φ_{PAL} is crucial. Outside this subspace, operations may generate configurations that do not satisfy PAL. Closure of the operator algebra is only required and only holds on Φ_{PAL}.

3.3 Bracket definition

For $O_i, O_j \in \mathcal{O}$, we define the commutator bracket

$$[O_i, O_j]\phi = O_i(O_j\phi) - O_j(O_i\phi), \qquad \phi \in \Phi_{\text{PAL}}. \tag{3}$$

This bracket is linear in each argument and satisfies the usual antisymmetry

$$[O_i, O_j] = -[O_j, O_i].$$

The Jacobi identity follows from the associativity of operator composition on Φ_{PAL}.

4 Closure Theorem

We now state and prove the main operator closure theorem.

Theorem 1 (Operator closure under PAL–coherence)**.** *Let $\mathcal{O} = \{\mathcal{T}, \mathcal{C}, \mathcal{R}, \mathcal{N}, \mathcal{G}\}$ be the PFT operator set acting on PAL–coherent configurations Φ_{PAL} on the AOL. Then for any pair $O_i, O_j \in \mathcal{O}$ there exist real coefficients $c^k_{ij}(p)$, depending on prime indices p and hierarchy indices (r, m, n, a, k), such that*

$$[O_i, O_j] = \sum_k c^k_{ij}(p)\, O_k \tag{4}$$

as operators on Φ_{PAL}. The coefficients are determined by convergent prime–indexed harmonic sums constrained by AOL spectral bounds.

Proof sketch. The proof proceeds in three parts.

Step 1: Domain preservation. By construction, each O_i maps Φ_{PAL} into itself. Therefore, for any $\phi \in \Phi_{\text{PAL}}$,

$$O_j\phi \in \Phi_{\text{PAL}},$$

and then

$$O_i(O_j\phi) \in \Phi_{\text{PAL}}.$$

Similarly $O_j(O_i\phi) \in \Phi_{\text{PAL}}$. Hence $[O_i, O_j]\phi$ is well–defined for all $\phi \in \Phi_{\text{PAL}}$.

Step 2: Local decomposition on the AOL. Each operator O_i is defined in terms of local actions on the AOL: differences along edges, curvature–weighted sums over faces, recursion steps on networks and so on. The action of O_i on a configuration can be decomposed into contributions from local patterns on the lattice.

On a fixed local pattern (for example, a block corresponding to the Lagrange–hex $\sqrt{1}$–$\sqrt{6}$ kernel), the composition O_iO_j can be expressed as a linear combination of the basic operations associated with $\mathcal{T}, \mathcal{C}, \mathcal{R}, \mathcal{N}, \mathcal{G}$, with coefficients depending on prime labels and hierarchy indices. This follows from the fact that the operator family is closed under composition at the level of local building blocks.

Step 3: PAL constraints and harmonic sums. PAL coherence restricts allowed compositions. In particular, contributions to $[O_i, O_j]$ that would violate PAL (by introducing non–neutral flux on prime faces) are cancelled or forbidden. The remaining contributions can be parameterised by prime–indexed harmonic sums of the form

$$c_{ij}^k(p) = \sum_{n \in \mathbb{P}} f_{ij}^k(n),$$

where $f_{ij}^k(n)$ are functions capturing phase–angle differences and curvature weights associated with prime–indexed faces. A typical example (studied in more detail in Section 5) is

$$c_{ij}^k(p) = \sum_{n \in \mathbb{P}} \frac{\sin(2\pi n \theta_{ij})}{n^s},$$

with $s = 1$ or slightly larger and θ_{ij} the PAL phase angle difference between sectors associated with O_i and O_j.

Spectral bounds on the AOL curvature operator ensure that these sums converge. The AOL spectrum is discrete and the curvature weights satisfy decay or boundedness conditions, so the contributions from large primes are limited. This yields finite $c_{ij}^k(p)$ for all relevant i, j, k.

Combining local decompositions with PAL constraints and convergence of the prime sums, one obtains the closure relation (4). □

Remark 1. *The theorem states closure on the PAL–coherent subspace. Outside that subspace the same algebraic relations need not hold, and operator compositions may generate non–physical configurations.*

5 Structure Constants and Prime Harmonics

The closure theorem expresses commutators in terms of structure constants $c_{ij}^k(p)$. This section describes their form and role.

5.1 Prime–indexed harmonic sums

A typical form for the structure constants is

$$c_{ij}^k(p) = \sum_{n \in \mathbb{P}} \frac{\sin(2\pi n \theta_{ij})}{n^s}, \tag{5}$$

where:

- $\mathbb{P}$ is the set of primes,
- θ_{ij} is a PAL phase angle difference characteristic of the operators O_i and O_j,
- $s \geq 1$ is an exponent determined by curvature weights and AOL spectral properties.

This form reflects the fact that:

- each prime–indexed face contributes a phase–dependent term to the commutator,
- PAL coherence suppresses contributions that do not satisfy flux neutrality,
- curvature weights produce denominators that grow with prime size.

5.2 Spectral boundedness

Convergence of (5) relies on spectral bounds. Let λ_p be curvature eigenvalues associated with prime–indexed faces. Under appropriate boundedness conditions on λ_p and smoothness in θ_{ij}, the sum converges either absolutely or conditionally, depending on s and the distribution of primes.

In PFT, the AOL spectrum and PAL structure are chosen so that these sums are finite. This ensures that:

- structure constants are well–defined,
- operator norms are finite on Φ_{PAL},
- the algebra is stable under repeated commutations.

5.3 Example: $[\mathcal{R}, \mathcal{N}]$[R,N] to $\mathcal{G}$G

As an illustrative example, consider $\mathcal{R}$ acting on network m and $\mathcal{N}$ coupling networks m and n. One can write

$$\mathcal{R}_m \mathcal{N}_{mn} \phi = \mathcal{N}_{mn} \mathcal{R}_n \phi + B_{mn} \phi, \tag{6}$$

where B_{mn} is a boundary term encoding flux imbalances at network interfaces. PAL coherence requires that such imbalances be re–expressed in terms of global adjustments, which are collected into $\mathcal{G}$. Thus

$$[\mathcal{R}, \mathcal{N}] = \mathcal{R}_m \mathcal{N}_{mn} - \mathcal{N}_{mn} \mathcal{R}_m = c^{\mathcal{G}}_{\mathcal{R}\mathcal{N}}(p)\, \mathcal{G}, \tag{7}$$

with

$$c^{\mathcal{G}}_{\mathcal{R}\mathcal{N}}(p) = \sum_{p \in \mathbb{P}} \frac{\sin\left(2\pi p \, \frac{m-n}{m+n}\right)}{p}. \tag{8}$$

The phase factor encodes the relative alignment of networks m and n under PAL; the denominator reflects the prime weighting.

6 Lagrange–Hex Ghost Kernel and Invariance

The Lagrange–hex $\sqrt{1}$–$\sqrt{6}$ ghost kernel plays a special role in the operator algebra.

6.1 Ghost kernel

The ghost kernel is the minimal set of displacement classes

$$\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}\}$$

in the Lagrange–hex projection, together with their adjacency, curvature and phase profiles. It defines a canonical local block from which larger structures on the AOL are built.

6.2 Operator action on the kernel

Each operator in $\mathcal{O}$ has a well–defined action on configurations restricted to the ghost kernel. For example:

- $\mathcal{T}$ moves pattern content between sites related by these displacements,
- $\mathcal{C}$ computes curvature–weighted differences around hexagonal loops,
- $\mathcal{R}$ iterates kernel–level patterns along hierarchy indices,
- $\mathcal{N}$ couples kernels associated with different primes or networks,
- $\mathcal{G}$ aggregates kernel actions into coarse–grained evolution.

6.3 Invariance under closure

Because the closure theorem is established locally and extended across the lattice, the ghost kernel is invariant under the commutator algebra. That is, when restricted to the kernel, the set of operators produced by commutators remains within the span of $\mathcal{T}, \mathcal{C}, \mathcal{R}, \mathcal{N}, \mathcal{G}$.

This invariance means the ghost kernel can serve as a testbed for analysing representations of the algebra and for constructing effective field descriptions.

7 Consequences for Representation Theory and Unification

Operator closure has several structural consequences.

7.1 Finite–dimensional Lie–type structure

On Φ_{PAL}, the commutator bracket gives $\mathcal{O}$ the structure of a finite–dimensional Lie–type algebra with:

- a finite basis of generators $\mathcal{T}, \mathcal{C}, \mathcal{R}, \mathcal{N}, \mathcal{G}$,
- structure constants $c_{ij}^{k}(p)$ encoding PAL and AOL information,
- closure under commutators,
- satisfaction of the Jacobi identity.

The algebra is "Lie–type" because it shares the core features of Lie algebras but is built on discrete, prime–indexed operators rather than continuous symmetry generators.

7.2 Representations as physical sectors

Physical sectors correspond to representations of this algebra:

- curvature–dominated representations yield GR–like behaviour,
- cascade–dominated representations yield QFT–like behaviour,
- mixed representations describe coupled gravity–field configurations.

Patterns on the AOL that transform irreducibly under this algebra correspond to fundamental dynamical sectors of PFT. Composite sectors correspond to reducible representations.

7.3 Unification

Because GR and QFT sectors both arise as representations of the same operator algebra on the same substrate, PFT unifies them at the algebraic level. There is no need to introduce distinct, incompatible operator sets for gravity and gauge fields; both are realised through different representations of $\mathcal{O}$ acting on PAL–coherent states.

8 Discussion and Outlook

We have shown that the core PFT operator hierarchy forms a closed algebra on PAL–coherent configurations on the Allen Orbital Lattice. The closure is controlled by prime–indexed harmonic structure constants and is compatible with the discrete, prime–indexed geometry of the AOL and with PAL constraints.

Key points include:

- definition of the operator set $\mathcal{O} = \{\mathcal{T}, \mathcal{C}, \mathcal{R}, \mathcal{N}, \mathcal{G}\}$ and their shared domain Φ_{PAL},
- commutator bracket with closure expressed in terms of structure constants $c_{ij}^{k}(p)$,
- convergence of prime–indexed harmonic sums due to AOL spectral bounds,
- invariance of the Lagrange–hex $\sqrt{1}$–$\sqrt{6}$ ghost kernel under the algebra,
- interpretation of GR and QFT sectors as representations of the same discrete operator algebra.

Future work may refine this picture by:

- classifying irreducible representations of $\mathcal{O}$ and mapping them to physical sectors,
- computing explicit commutators in specific hierarchy sectors (r, m, n, a, k),
- analysing how operator closure interacts with renormalization group flows in the PFT hierarchy,
- exploring whether the algebra admits central extensions or deformations relevant for anomalies or quantum corrections.

The main conclusion is that Pattern Field Theory possesses an internally consistent operator framework. The closure of its operator algebra on PAL–coherent states provides a robust structural foundation for the unified description of curvature, cascades and fields.

Appendix A — Glossary of Terms

Pattern Field Theory (PFT) Unified framework in which all structure and dynamics are described as patterns evolving on the Allen Orbital Lattice under Phase Alignment Lock constraints.

Allen Orbital Lattice (AOL) Prime–indexed orbital–curvature lattice carrying sites, edges, faces, curvature weights, phase data and recursion structure. It is the discrete substrate in PFT.

Phase Alignment Lock (PAL) Coherence condition requiring exact flux neutrality on all prime–indexed faces of the AOL. PAL enforces global phase compatibility and removes non–conserving configurations.

PAL–coherent configuration Configuration ϕ on the AOL that satisfies PAL on every prime–indexed face. Operators in $\mathcal{O}$ are defined so that they map PAL–coherent configurations to PAL–coherent configurations.

Transport operator $\mathcal{T}$T Operator that implements local movement of pattern content along edges of the AOL, respecting curvature and phase data.

Curvature–weighted derivative $\mathcal{C}$C Operator that computes discrete derivatives of pattern fields, weighted by local curvature contributions on faces.

Recursion operator $\mathcal{R}$R Operator that advances patterns along recursion depth indices, implementing hierarchical constructions and repeated updates.

Cross–network coupling operator $\mathcal{N}$N Operator that couples pattern configurations on different prime–indexed networks or sectors.

Global evolution operator $\mathcal{G}$G Operator that aggregates local actions into larger–scale evolution, capturing the net effect of $\mathcal{T}, \mathcal{C}, \mathcal{R}, \mathcal{N}$ across regions of the AOL.

Lagrange–hex projection Representation of the AOL that organises minimal displacement modes into hexagonally structured layers labelled by distances $\sqrt{n}$.

Ghost kernel Minimal set of displacement classes $\{\sqrt{n} \mid n = 1, \ldots, 6\}$ and their local curvature and adjacency profiles, used as a canonical block for analysing local operator action.

Structure constants Coefficients $c_{ij}^k(p)$ appearing in the closure relation $[O_i, O_j] = \sum_k c_{ij}^k(p)\, O_k$, determined by prime–indexed harmonic sums and AOL spectral properties.

Appendix B — PFT Internal Bibliography

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Appendix C — Notes on Algebraic Structure

C.1 Jacobi identity

On Φ_{PAL}, operator composition is associative. Together with the antisymmetry of the commutator, this implies the Jacobi identity

$$[O_i, [O_j, O_k]] + [O_j, [O_k, O_i]] + [O_k, [O_i, O_j]] = 0$$

for all $O_i, O_j, O_k \in \mathcal{O}$. This confirms the Lie–type structure of the algebra.

C.2 Rosser–type confluence

The term "Lie–Rosser" reflects the fact that different sequences of operator applications leading from a given initial configuration to a final PAL–coherent configuration yield the same result modulo elements of the operator algebra. This is analogous to confluence properties in rewriting systems and ensures that evolution is well–defined at the algebraic level.

C.3 Extensions and deformations

One can consider extensions of the algebra in which:

- additional operators are added to capture new sectors,
- structure constants are deformed to reflect quantum corrections or anomaly–like effects.

Such deformations must preserve PAL coherence and AOL spectral bounds to remain physically meaningful in PFT.

Document Timestamp and Provenance

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