PFT Hierarchy as Renormalization Group: Multi–Scale Flow on the Allen Orbital Lattice

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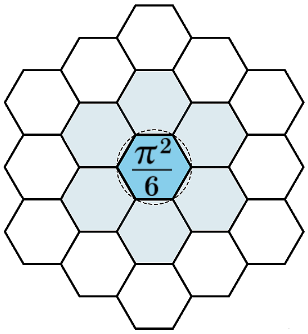

Abstract

Pattern Field Theory (PFT) organises dynamics on the Allen Orbital Lattice (AOL) through a hierarchy indexed by (r, m, n, a, k). The scale index r tracks coarse–graining, m and n label prime–indexed network components, and a and k describe recursion depth and cascade layering. This hierarchy governs how patterns evolve from ultraviolet (UV) conservation regimes to infrared (IR) effective field theories such as general relativity (GR) and quantum field theory (QFT).

In this paper we formalise the PFT hierarchy as a renormalization group (RG) flow on the AOL. Coarse–graining corresponds to prime–composite merging in the Lagrange–hex projection of the $\sqrt{1}$–$\sqrt{6}$ ghost system. PAL–stable configurations appear as fixed points of the flow. We show that the UV limit is controlled by the convergence of the prime–weighted curvature sum related to $\zeta(2) = \pi^2/6$, while the IR limit yields the emergent GR and QFT sectors described in previous PFT work.

The resulting RG structure provides a single multi–scale flow that connects discrete AOL curvature, PAL coherence, emergent spacetime, and effective field interactions. It replaces continuum ultraviolet divergences with finite combinatorial flows on a prime–indexed lattice and supplies a collapse taxonomy for how structures approach fixed points under coarse–graining.

1 Introduction

Renormalization group (RG) ideas are central to contemporary physics. They describe how effective couplings, degrees of freedom and observables change as one varies the scale at which a system is probed. In conventional quantum field theory (QFT), RG flows control how couplings run with energy and how fixed points organise phases of matter.

Pattern Field Theory (PFT) introduces a discrete substrate for unification: the Allen Orbital Lattice (AOL), a prime–indexed orbital–curvature lattice with Phase Alignment Lock (PAL) enforcing flux neutrality on prime–labelled faces. Dynamics are implemented through event cascades constrained by PAL. General relativity (GR) and the Standard Model sector of QFT emerge as infrared projections of this substrate.

From the perspective of PFT, the question is not whether an RG structure exists, but how it is encoded in the discrete hierarchy that organises patterns on the AOL. PFT already distinguishes between different scales, recursion depths and cascade layers using indices (r, m, n, a, k). The purpose of this paper is to formalise that hierarchy as a renormalization group flow and to identify its fixed points and collapse regimes.

We proceed as follows. Section **??** recalls the PFT hierarchy indices and their roles. Section **??** describes the mapping between PFT hierarchy and RG ideas. Section **??** introduces an explicit beta function for PAL couplings and identifies fixed points. Section **??** discusses coarse–graining in terms of the Lagrange–hex $\sqrt{1}$–$\sqrt{6}$ unit cell. Section **??** provides a collapse taxonomy in terms of the scale index r. Section **??** connects the RG flow to emergent GR and QFT. Section **??** compares this structure with standard Wilsonian RG. Section **??** summarises implications. Appendices contain a glossary, internal PFT bibliography and additional mathematical remarks.

2 PFT Hierarchy and Indices

The PFT hierarchy assigns discrete and continuous indices to patterns on the Allen Orbital Lattice. This section recalls the main indices and their interpretation.

2.1 Scale index r

The index $r \in \mathbb{R}_{\geq 0}$ measures the coarse–graining scale. Small r corresponds to fine–grained descriptions that resolve detailed AOL structure and individual ghost–layer interactions. Large r corresponds to increasingly coarse descriptions in which local details are averaged over and only aggregated quantities remain.

Operationally, increasing r corresponds to grouping AOL cells (sites, edges, faces) into blocks, combining prime–labelled faces into effective units and replacing local features by effective couplings. In this sense, r plays the role of an RG scale parameter.

2.2 Network indices m and n

The indices m and n label prime–indexed networks or sectors within the AOL. For example:

- one network might be associated with a particular prime factorisation pattern,
- another might encode a distinct curvature or phase–assignment motif.

Interactions between networks are mediated by cross–network operators and cascades. The prime nature of m and n reflects the prime indexing of AOL faces and the modular structure of PAL constraints.

2.3 Recursion index a

The index $a \in \mathbb{Z}_{\geq 0}$ measures recursion depth in event cascades or field constructions. At $a = 0$, a pattern may be given by a base configuration; higher a correspond to repeated applications of operators, iterative refinements or embedded cascades.

Recursion depth controls how far a pattern has been developed from a minimal seed under PFT evolution rules. RG flows often track how effective descriptions change under successive coarse–grainings; the recursion index in PFT is the discrete counterpart of repeated RG transformations.

2.4 Cascade index k

The index $k \in \mathbb{Z}_{\geq 0}$ labels layers in a cascade. A cascade can be organised into levels representing successive branching events. The index k distinguishes:

- early–stage events close to the initial configuration,
- intermediate branchings,
- late–stage events near final states.

In the RG picture, different k can correspond to different effective scales at which interactions are resolved or integrated out.

2.5 Summary

Together, (r, m, n, a, k) capture:

- the coarse–graining scale (r),
- prime–sector labels (m, n),
- recursion structure (a),
- cascade layering (k).

These indices provide a natural coordinate system for describing multi–scale dynamics on the AOL. In the following sections, r is promoted to a genuine RG scale, with m, n, a, k specifying the sector in which a coupling is defined.

3 RG Interpretation of the PFT Hierarchy

Renormalization group flows describe how effective couplings change with scale. In the PFT setting, couplings depend on r and are constrained by PAL on the AOL. This section explains the mapping.

3.1 Effective couplings on the AOL

Consider a PAL–coherent configuration of curvature and phase assignments on the AOL. One can define effective couplings $g(r; m, n, a, k)$ that parameterise:

- the strength of interactions between networks labelled by m and n,
- the impact of recursion depth a on local dynamics,
- the contribution of cascade layer k,

- the effect of coarse–graining at scale r.

As r increases, the detailed structure of the AOL is averaged over larger blocks. The couplings g adjust to capture the net effect of fine–grained PAL–coherent configurations that have been integrated out.

3.2 Coarse–graining and prime–composite merging

A characteristic feature of PFT is the prime–indexed structure of the AOL. Faces are labelled by primes, and PAL constraints operate on prime–labelled cells. Coarse–graining then naturally involves prime–composite merging:

- clusters of faces corresponding to composite labels can be treated as effective prime–like units,
- repeated application of this procedure leads to a hierarchy of effective labels.

In the Lagrange–hex projection, this process can be visualised as merging $\sqrt{n}$ layers into effective displacement scales. For example, $\sqrt{4} = 2\sqrt{1}$ can be represented as an effective $\sqrt{1}^*$ unit after sufficient coarse–graining.

3.3 RG flow on the PFT hierarchy

With these ingredients, an RG flow in PFT can be defined as the evolution of $g(r; m, n, a, k)$ under changes in r that implement prime–composite merging and ghost–layer averaging. The indices (m, n, a, k) specify the sector; the flow equation describes how couplings in that sector are driven toward or away from fixed points as r varies.

4 Beta Function and Fixed Points

We now introduce a beta function for PAL couplings and discuss fixed points.

4.1 Definition of the beta function

Let $g(r)$ denote a generic PAL–compatible coupling at scale r in a given sector (m, n, a, k). The beta function $\beta(g)$ expresses how g changes with r:

$$\beta(g) = \frac{\mathrm{d}g}{\mathrm{d}\log r}. \tag{1}$$

In PFT, the discrete structure and the prime–indexed nature of the AOL suggest a particular form for the flow near prime–related fixed points.

Theorem 1 (PFT RG flow)**.** *For a PAL coupling $g(r)$ associated with a prime index p, the beta function near the fixed point takes the form*

$$\beta(g) = (r - r^*)\frac{\partial g}{\partial r}, \quad r^* = \log p, \tag{2}$$

where r^* is the fixed point scale corresponding to the prime p.

Proof sketch. The dependence of g on r arises from coarse–graining. At a scale r, one considers blocks that aggregate AOL cells over a characteristic size related to e^r. Prime–indexed structures with label p are naturally stabilised when the block scale matches the logarithm of p, leading to $r^* = \log p$.

Near r^*, write $g(r) = g^* + \delta g(r)$ with g^* a fixed point value. A linear expansion in $(r - r^*)$ produces

$$\beta(g) = \frac{\mathrm{d}g}{\mathrm{d}\log r} \approx (r - r^*) \frac{\partial g}{\partial r}\Big|_{r=r^*},$$

which yields the stated form. PAL coherence ensures that g^* is stable: deviations in g that would violate PAL are suppressed under the flow. □

4.2 UV and IR behaviour

The UV and IR limits of the flow can now be characterised.

- As $r \to \infty$, the flow probes very coarse scales. Prime contributions are integrated into an effective sum. The relevant aggregate quantity is a prime–weighted curvature sum whose convergence is controlled by a zeta–like structure related to $\zeta(2) = \pi^2/6$. This yields a UV fixed point dominated by conservation and PAL neutrality.
- As $r \to 0$, the flow resolves fine–grained detail on the AOL. Couplings approach configurations controlled by local PAL constraints and discrete curvature patterns. Infrared limits in particular sectors yield effective GR and QFT descriptions.

Remark 1. *The appearance of $\zeta(2)$ reflects the role of prime–indexed harmonic sums in earlier PFT work. The RG perspective clarifies that this structure governs the UV behaviour of couplings under coarse–graining.*

5 Lagrange–Hex Coarse–Graining Unit

The Lagrange–hex projection provides a natural coarse–graining unit cell for PFT RG flows.

5.1 Ghost layers and displacement classes

In the Lagrange–hex representation, minimal displacement modes are grouped into layers labelled by $\sqrt{n}$, with n a positive integer. The first six layers $\sqrt{1}, \dots, \sqrt{6}$ define a ghost kernel that captures the minimal local structure from which larger–scale patterns are built.

Each layer encodes:

- adjacency relations among sites at a given displacement scale,
- local curvature increments,
- PAL–compatible phase assignments.

5.2 Unit cell for RG transformations

Coarse–graining in PFT can be implemented by taking the $\sqrt{1}$–$\sqrt{6}$ block as an RG unit cell. A single RG step then consists of:

- identifying such blocks across the AOL,
- averaging or integrating out fine–grained details within each block,
- producing effective couplings that summarise the block's influence on larger–scale behaviour.

For example, consider the relation

$$\sqrt{4} = 2\sqrt{1}. \tag{3}$$

Within a block, repeated displacements of size $\sqrt{1}$ can mimic a single displacement of size $\sqrt{4}$. Under coarse–graining these become indistinguishable: their effects on PAL–constrained fluxes and curvature can be represented by an effective $\sqrt{1}^*$ layer with adjusted couplings.

5.3 Iterated coarse–graining

Successive RG steps repeatedly apply this block–averaging process. The ghost kernel remains structurally invariant under PAL–diffeomorphisms, as discussed in earlier work on completion equivalence. What changes are the effective couplings assigned to blocks, which follow the beta function equation described in Section **??**.

6 Collapse Taxonomy

RG flows are often characterised in terms of fixed points and the behaviour of trajectories near them. In PFT, the scale index r and PAL constraints lead to a collapse taxonomy.

6.1 Subcritical, critical and ultra–sparse regimes

Define a characteristic scale r_c which marks a transition between regimes dominated by local discrete structure and those dominated by global conservation.

- **Subcritical regime** ($r < r_c$): Fine–grained AOL structure and individual ghost–layer interactions are directly relevant. Couplings are sensitive to local combinatorics. Cascades can explore detailed patterns before being damped by PAL.
- **Critical regime** ($r \approx r_c$): Local and global effects balance. RG flows are slow; small changes in r can reorganise large–scale behaviour. In this regime, scale–invariant features and fractal–like cascades appear naturally.
- **Ultra–sparse regime** ($r > r_c$): Coarse–graining has integrated out most local details. Effective couplings approach UV fixed points dominated by conservation; many cascades collapse rapidly to PAL–neutral configurations with minimal structure.

6.2 Collapse and universality

Under repeated coarse–graining, many different microscopic configurations flow toward the same effective couplings. This is the origin of universality: distinct initial AOL patterns that share coarse features and satisfy PAL at the same scale can produce the same infrared physics.

Collapse in PFT refers to the tendency of flows to converge on a small set of fixed points or attractor manifolds in coupling space. The RG hierarchy provides a classification:

- flows that end in GR–like curvature sectors,
- flows that end in QFT–like cascade sectors,
- flows that remain in conservation–dominated sectors.

7 UV and IR Limits: Connection to GR and QFT

The PFT RG structure links the discrete AOL substrate to emergent GR and QFT.

7.1 UV fixed point and conservation–dominated regime

As $r \to \infty$, coarse–graining replaces detailed AOL structure by aggregate quantities controlled by prime–indexed sums. PAL ensures that only flux–neutral configurations survive. The convergence of prime–weighted curvature sums related to $\zeta(2) = \pi^2/6$ sets a UV fixed point in which:

- local fluctuations have been averaged out,

- only conservation–consistent modes remain,
- couplings approach constant values determined by discrete number–theoretic structure.

This regime is not populated by ordinary fields; it describes the deepest conservation structure underlying the emergent theories.

7.2 IR limits and emergent GR

In the curvature sector, flows that move toward smaller r and larger recursion depth a can arrange PAL–coherent curvature patterns that, when projected to the continuum, satisfy Einstein–like equations. The metric emerges as an effective phase–gradient of pattern fields, and curvature tensors derive from discrete fluxes on the AOL.

The RG perspective clarifies that GR appears in a particular IR corner of the PFT hierarchy:

- r is small enough to retain geometric detail but large enough that local combinatorial noise has been smoothed,
- couplings have flowed to a region where continuum approximations and differential equations are valid,
- PAL constraints manifest as continuum conservation laws and Bianchi identities.

7.3 IR limits and emergent QFT

In the cascade sector, flows that emphasise event trees and branching rather than static curvature produce effective quantum field descriptions. As detailed in earlier work, gauge groups and scattering amplitudes arise as infrared shadows of PAL–constrained cascades.

In RG language:

- r controls how many cascade layers are integrated into effective vertices and propagators,
- couplings encode the probability weights of different branching patterns,
- fixed points correspond to scale–invariant interaction structures, such as renormalizable QFTs.

The same AOL and PAL substrate thus yields both gravity and QFT sectors through different RG trajectories in the PFT hierarchy.

8 Comparison with Wilsonian RG

It is useful to compare the PFT RG picture with the standard Wilsonian RG in continuum QFT.

8.1 Degrees of freedom

In Wilsonian RG, one integrates out high–momentum modes of fields defined on a continuum spacetime. In PFT, one integrates out fine–grained AOL structure and local cascade details. The role of momentum shells is played by blocks of ghost–layer displacements and prime–indexed faces.

8.2 Coupling space

Both approaches define a space of couplings and flow equations for these couplings. In PFT, coupling space is structured by the indices (r, m, n, a, k) and by PAL constraints. Only PAL–compatible directions in coupling space are allowed; others are suppressed or forbidden.

8.3 Fixed points and universality

Fixed points and universality classes exist in both frameworks. In PFT, UV and IR fixed points are governed by discrete combinatorics, prime structures and PAL coherence. Universality arises from the collapse of many microscopic configurations onto the same trajectory in coupling space under coarse–graining.

8.4 Background independence

A key difference lies in background independence. Wilsonian RG is usually formulated on a fixed background spacetime. PFT implements RG flow on a discrete lattice that is itself subject to completion equivalence and PAL–diffeomorphisms. Background independence is built in at the substrate level.

9 Discussion and Outlook

We have recast the PFT hierarchy as a renormalization group flow on the Allen Orbital Lattice. The scale index r controls coarse–graining, prime–composite merging and ghost–layer averaging. Couplings $g(r)$ obey a beta function whose fixed points are associated with prime–related scales and PAL–coherent configurations. UV behaviour is governed by conservation–dominated sums related to $\zeta(2)$; IR behaviour yields emergent GR and QFT.

Key structural points include:

- the use of the Lagrange–hex $\sqrt{1}$–$\sqrt{6}$ block as an RG unit cell,
- the collapse taxonomy in terms of subcritical, critical and ultra–sparse regimes,
- the identification of PAL–compatible flows that preserve discrete conservation laws,
- the integration of gravity and field theory sectors into a single RG hierarchy.

Future work may extend this analysis by:

- computing explicit flows for particular sectors (m, n, a, k),
- studying how completion equivalence restricts allowed RG trajectories,
- exploring whether the PFT RG structure imposes quantitative constraints on observable couplings in the Standard Model or on cosmological parameters,
- analysing the interplay between RG flows and cross–coherent cascade phenomena in CCCT.

The main conclusion is that Pattern Field Theory does not only unify GR and QFT at a static level. It also provides a unified renormalization group framework in which the multi–scale flow of couplings is rooted in prime–indexed discrete geometry and PAL constraints on the Allen Orbital Lattice.

Appendix A — Glossary of Terms and Indices

Pattern Field Theory (PFT) Unified framework in which all structure and dynamics are described as patterns evolving on the Allen Orbital Lattice under Phase Alignment Lock constraints.

Allen Orbital Lattice (AOL) Prime–indexed orbital–curvature lattice carrying sites, edges, faces, curvature weights, phase data and recursion structure. It is the discrete substrate in PFT.

Phase Alignment Lock (PAL) Coherence condition requiring exact flux neutrality on all prime–indexed faces of the AOL. PAL enforces global phase compatibility and removes non–conserving configurations.

PFT hierarchy indices The tuple (r, m, n, a, k) used to organise multi–scale dynamics in PFT:

- r — coarse–graining scale index,
- m, n — prime–indexed network labels,
- a — recursion depth index,
- k — cascade layer index.

Renormalization group (RG) Framework describing how effective couplings depend on scale. In PFT, RG flows are realised as changes in $g(r; m, n, a, k)$ under coarse–graining of the AOL.

Beta function Function $\beta(g) = \mathrm{d}g/\mathrm{d}\log r$ governing the scale dependence of a coupling $g(r)$.

Fixed point Value g^* such that $\beta(g^*) = 0$. In PFT, fixed points correspond to PAL–stable configurations that are invariant under coarse–graining at a particular scale.

Lagrange–hex projection Representation of the AOL that organises minimal displacement modes into hexagonally structured layers labelled by distances $\sqrt{n}$.

Ghost layer Structured pattern of allowed lattice moves at a fixed displacement scale $\sqrt{n}$ in the Lagrange–hex projection.

Ghost kernel Minimal set of displacement classes, such as $\{\sqrt{n} \mid n = 1, \ldots, 6\}$, and their local curvature and adjacency profiles, used as a canonical block for coarse–graining.

Subcritical, critical, ultra–sparse regimes Regimes of RG flow characterised by the value of the scale index r relative to a characteristic scale r_c, controlling the balance between local detail and global conservation.

General relativity (GR) Classical field theory of spacetime curvature described by the Einstein equations. In PFT, GR arises as an infrared projection of PAL–constrained curvature dynamics.

Quantum field theory (QFT) Framework describing particles and interactions as excitations of fields on a spacetime background. In PFT, QFT arises as an infrared projection of PAL–constrained cascades on the AOL.

Appendix B — PFT Internal Bibliography

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Appendix C — Notes on RG Flow Structure in PFT

C.1 Completion equivalence and RG

Completion equivalence of AOL structures implies that RG flows must be defined on equivalence classes of completions. PAL–diffeomorphisms relate completions without changing physical observables, so RG trajectories are traced in a quotient space where redundant degrees of freedom have been identified.

C.2 Discrete vs continuum RG

Although PFT RG flows operate on discrete lattices, infrared projections can be compared directly to continuum RG flows. The mapping from discrete r to continuum momentum scales is mediated by effective metrics and field configurations induced by PAL–coherent curvature.

C.3 Constraints from PAL

PAL restricts the directions in which couplings can flow. Couplings that would lead to PAL violations are dynamically suppressed or excluded from the physically relevant coupling space. This provides a structural explanation for why certain pathological behaviours do not arise in PFT RG flows.

Document Timestamp and Provenance

This paper forms part of the dated Pattern Field Theory research chain beginning May 2025, recorded through server logs, cryptographic hashes and versioned texts on PatternFieldTheory.com, establishing priority and authorship continuity for the renormalization group formulation of the PFT hierarchy on the Allen Orbital Lattice.