Logarithms as Geometric Charts: From Riemann Surfaces to Discrete Lattice Lifting with Applications to pH, Decibel Scales, Magnitude Scales, and Orbital Lifts

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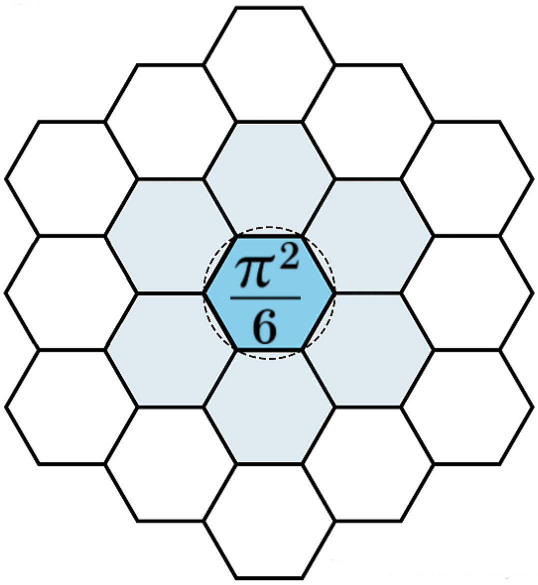

Abstract

Logarithms appear throughout mathematics and the natural sciences as the canonical means of converting multiplicative structure into additive coordinates. In complex analysis, this role becomes structural: the complex logarithm requires a lift of the plane to a multi-sheeted space, classically realized as a Riemann surface.

In this paper we reinterpret logarithms as geometric chart maps that lift ratio-based domains into extended coordinate spaces, and we show how this single geometric idea unifies the complex logarithm with familiar logarithmic measurement scales such as pH, decibels, and magnitude scales, as well as with spiral and orbital representations of multiplicative progression.

For each domain, we define a corresponding three-dimensional lifted representation (surface, ramp, or helix) and reference reproducible generation procedures via external implementations. Finally, we contrast the classical, analytic construction of the logarithmic lift with a discrete, constructive realization on the Allen Orbital Lattice (AOL), where the lifted dimension is implemented as explicit structural state rather than as a purely representational covering space.

Introduction

Logarithms occupy a special position in both mathematics and the empirical sciences. They provide the natural coordinate system for quantities whose meaningful composition law is multiplicative rather than additive. This is why logarithmic scales appear in such diverse settings as the pH scale in chemistry, the decibel scale in signal processing, and magnitude scales in astronomy and geophysics.

In complex analysis, the logarithm plays a deeper and more structural role. The complex logarithm is not single-valued on the punctured plane, and its consistent definition requires lifting the domain to a multi-sheeted covering space. This construction, realized classically as the Riemann surface of the logarithm, is one of the canonical examples of how multivaluedness forces an extension of the coordinate domain into a higher-dimensional structure.

Traditionally, these two appearances of logarithms are treated as conceptually separate. In this paper we show that they are instances of the same underlying geometric operation: the use of the logarithm as a chart map that lifts a multiplicative domain into an additive, extended coordinate space.

The Classical Logarithmic Lift and the Riemann Surface

In classical complex analysis, the logarithm is defined as

$$\log z = \ln r + i(\theta + 2\pi k), \quad k \in \mathbb{Z},$$

where $z = re^{i\theta}$. The multivaluedness arises from the angular coordinate θ, which is defined only modulo 2π.

To obtain a single-valued object, one introduces the Riemann surface of the logarithm. This surface consists of a countable stack of copies of the punctured complex plane, glued along a chosen branch cut. Analytic continuation around the origin increments the sheet index k, producing a helical or stair-stepped topology.

A fully reproducible computational realization of this classical lift object is provided at:

patternfieldtheory.com/code/logarithmic-dimensional-shift-replication.php

The associated theoretical framework is presented in:

patternfieldtheory.com/papers/logarithmic-dimensional-shift.pdf

Logarithmic Scales as Geometric Chart Maps

Many familiar logarithmic scales in science can be understood as coordinate charts on multiplicative domains. The pH scale, the decibel scale, and magnitude scales all convert ratio spaces into additive coordinates. Cascaded ratios become linear translations in logarithmic space.

This is not merely numerical convenience. It is a change of coordinate geometry: the logarithm replaces a multiplicative composition law with an additive one.

Example Domains and Generated Lift Diagrams

In this section we outline several representative domains and the corresponding three-dimensional lifted geometries. For each case, the figures are intended to be generated procedurally by external reference implementations.

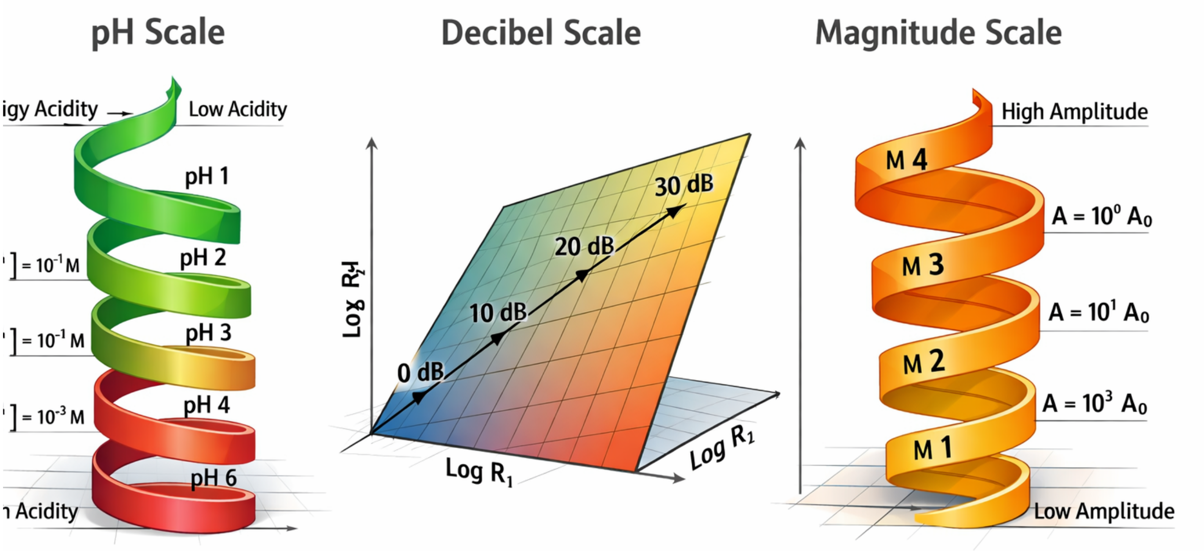

Figure 1: Unified geometric lift representations of logarithmic scales. Top: classical logarithmic helicoid / Riemann surface style lift. Middle: representative logarithmic scale lifts (e.g. pH, decibel, magnitude) shown as helical or ramped chart spaces. Bottom: spiral / orbital logarithmic lift illustrating combined cyclic phase and multiplicative scaling. These figures visualize the common geometric structure underlying logarithmic coordinates as lifts from multiplicative domains into additive chart spaces.

pH: Decade Lifting as a Helical Ramp

The pH scale is defined by

$$\mathrm{pH} = -\log_{10}[\mathrm{H}^+].$$

Successive decades of concentration can be represented as successive turns of a helix, with height proportional to the pH value.

Decibels: Ratio Space as an Additive Lift

For power ratios:

$$L = 10\log_{10}\left(\frac{P}{P_0}\right).$$

Cascaded gains become additive translations on a lifted surface or stepped ramp geometry.

Magnitude Scales: Amplitude and Energy Lifting

Magnitude scales in astronomy and seismology can be represented as stacked or helical lift surfaces, making explicit that the magnitude coordinate is a chart on a ratio space.

Orbits and Spiral Lifts

When cyclic motion is combined with multiplicative scaling, the natural geometry is a spiral or helix. Angular progression corresponds to phase, while vertical displacement corresponds to logarithmic change in scale.

Orbital Lift as a Logarithmic Helix

A particularly important unified geometric form is the *orbital lift*, in which cyclic phase progression is combined with multiplicative scaling to produce a smooth three-dimensional lifted structure.

A canonical continuous model is given by a helix with logarithmically growing radius. Let the phase parameter be $\theta \in [0, 10\pi]$. Define

$$r = e^{a\theta}, \quad z = \theta,$$

and embed in $\mathbb{R}^3$ by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = \theta.$$

This produces a smooth helical curve whose radius grows multiplicatively while height accumulates additively. The geometry is single-valued and continuous in three dimensions. However, projection to the xy-plane collapses the lifted state and produces a logarithmic spiral, in which multiple turns overlap in phase and seam-like scars appear.

To make this geometry concrete, consider the specific choice $a = 0.1$ and sample the curve at successive multiples of 2π, corresponding to full orbital turns. This produces exponential growth in radius combined with linear stacking in the lift coordinate z, directly illustrating the Logarithmic Dimensional Shift as a geometric process. In the Pattern Field Theory interpretation, r corresponds to ring depth, θ to phase (PAL-aligned), and z to accumulated dominion index combined with logarithmic scale.

θ	$r = e^{0.1\theta}$	$x = r\cos\theta$	$y = r\sin\theta$	z
0.00	1.00	1.00	0.00	0.00
6.28	1.87	1.87	0.00	6.28
12.57	3.51	3.51	0.00	12.57
18.85	6.59	6.59	0.00	18.85
25.13	12.36	12.36	0.00	25.13

This table shows explicitly how each full orbital turn produces multiplicative growth in radius while the lift coordinate accumulates additively, exactly as required by the logarithmic lifting principle.

Figure 2 illustrates this relationship between the lifted three-dimensional orbital geometry and its projected two-dimensional base representation.

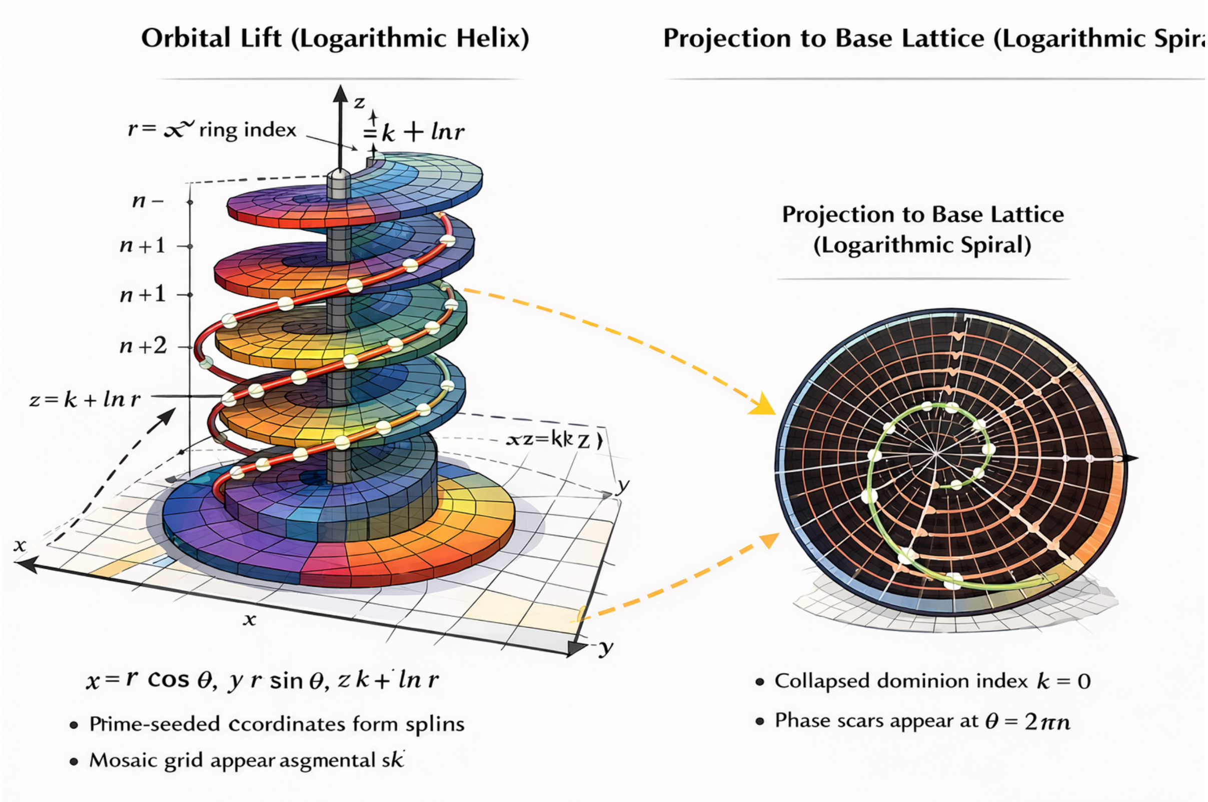

Figure 2: Orbital logarithmic lift geometry. Left: three-dimensional logarithmic helix combining cyclic phase and multiplicative radial scaling with additive vertical stacking ($z = k + \ln r$). Right: projection to the base lattice (or plane) collapses the dominion index and produces a logarithmic spiral in which phase seams (scars) appear at multiples of 2π. This illustrates how the lifted geometry is smooth and single-valued, while the projected representation reintroduces discontinuities.

Discrete Logarithmic Lifting on the Allen Orbital Lattice

In the Allen Orbital Lattice (AOL) framework, the substrate is discrete. The lifted dimension is implemented as explicit state, and the sheet index becomes an integer-valued field rather than an abstract analytic label.

Transport around a closed path produces a concrete increment of this state, and branch cuts become structural seams in the lattice.

Rigorous Example: Single-Valuedness of the Discrete Logarithm in the AOL Lift

Lemma 1. *The discrete logarithm on the Allen Orbital Lattice is single-valued when the sheet index is implemented as explicit structural state.*

Proof. In the Allen Orbital Lattice, let a site be described by discrete coordinates (q, r, s) together with an explicit sheet (dominion) index $k \in \mathbb{Z}$. Define the embedded coordinates by

$$X = q + \frac{r}{2}, \quad Y = \frac{\sqrt{3}}{2} r, \quad Z = k \ln \sqrt{q^2 + r^2 + s^2} + \beta\theta,$$

where θ is the angular coordinate induced by the prime-seeded lattice orientation.

Define the lifted discrete logarithm by

$$F(q, r, s, k) = \ln \sqrt{q^2 + r^2 + s^2} + i(\theta + 2\pi k).$$

Since the sheet index k is an explicit, integer-valued state variable and is not reduced modulo any period, the value of F is uniquely determined for each lattice state (q, r, s, k). Therefore, F is single-valued on the lifted lattice.

If one projects back to the base lattice by discarding the sheet index k, multiple distinct lifted states collapse to the same base coordinate (q, r, s), reintroducing discontinuities along seams. In the lifted structure, however, winding around the origin increments k additively and no discontinuity occurs.

Hence, seams are not present in the lifted structure itself but arise only under projection to a non-lifted representation. □

Remark 1. *This mirrors the role of branch cuts in the classical logarithmic Riemann surface. In the AOL formulation, the lift is not representational but constructive: dominion stacking replaces analytic continuation, and seams appear only when the lift state is discarded.*

Lemma 2 (Winding Increment and Seam-Free Transport)**.** *Let a closed transport loop in the base domain wind n times around the origin (or around the lattice-defined singular core). In the AOL lift with explicit sheet index $k \in \mathbb{Z}$, the resulting lifted transport satisfies*

$$k \mapsto k + n,$$

and the lifted logarithm value changes continuously along the lifted path, with no seam discontinuity.

Proof. In the classical logarithm, a single circuit around the origin changes the argument by $\theta \mapsto \theta + 2\pi$, which is the source of multivaluedness in the base domain. The lifted form records this explicitly by including the integer sheet parameter:

$$\log z = \ln r + i(\theta + 2\pi k), \quad k \in \mathbb{Z}.$$

A loop with winding number n adds $2\pi n$ to the angular coordinate, hence shifts the effective branch by $k \mapsto k + n$.

In the AOL formulation, the sheet index k is not an inferred analytic label but an explicit stored state. Therefore the transport update rule is constructive: each completed winding increments k by one, and n windings increment k by n. Because the lifted state changes by additive accumulation rather than by forced branch selection, the lifted path does not encounter a discontinuity. Discontinuities appear only if the projection discards k, collapsing distinct lifted states onto the same base site. □

Remark 2. *This lemma is the discrete analogue of path lifting in covering-space theory: the seam is not a feature of the lifted geometry, but of representing the lift on a single base sheet.*

Figure 3 shows representative lifted geometries for several logarithmic domains.

Representation Versus Construction

In the classical theory, the extra dimension is representational and lives in function space. In the AOL framework, the extra dimension is structural and lives in the substrate.

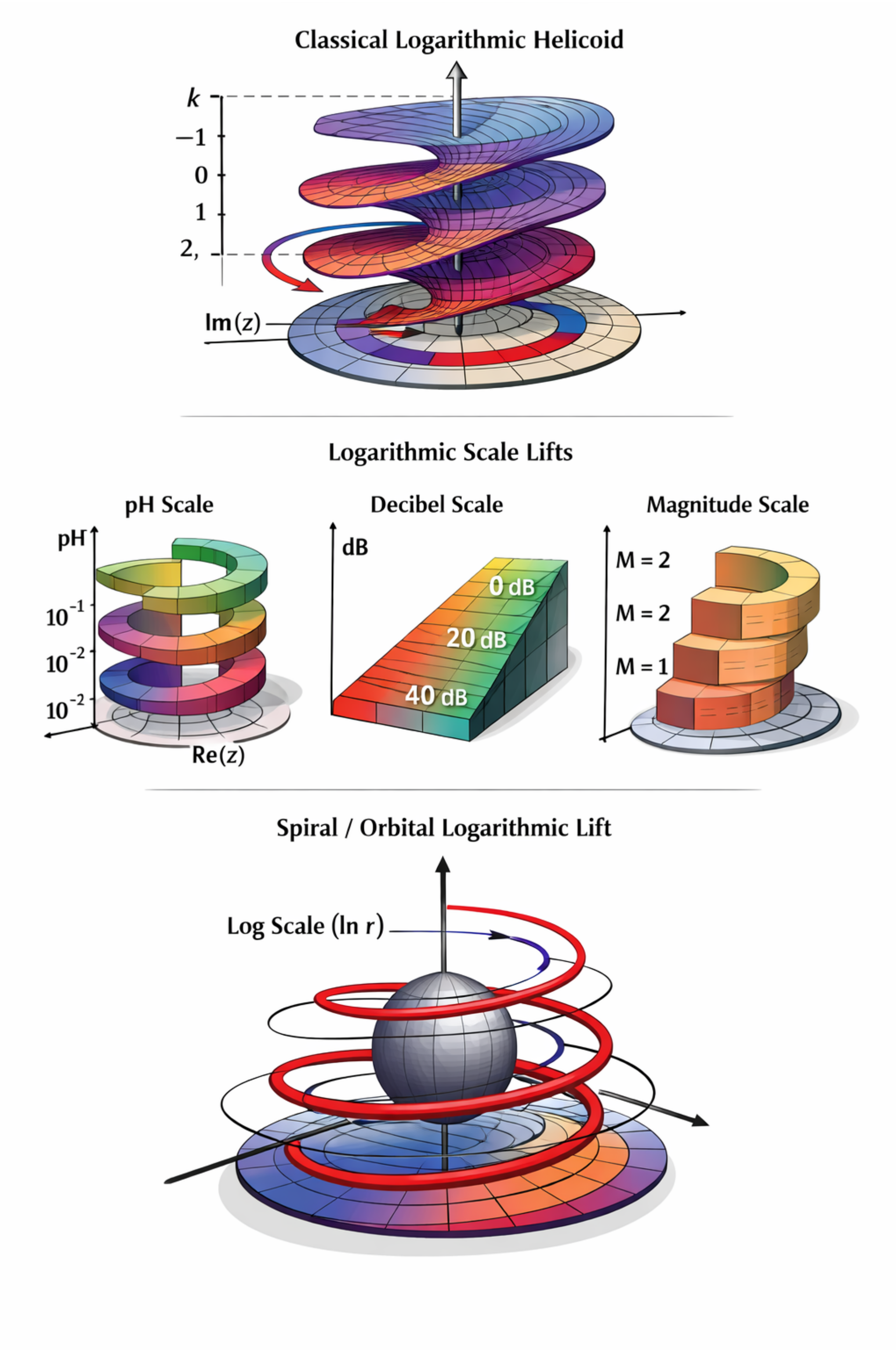

Figure 3: Unified geometric lift representations of logarithmic scales. Top: classical logarithmic helicoid (Riemann surface lift). Middle: pH, decibel, and magnitude scales represented as helical or ramp-like lifts. Bottom: spiral / orbital logarithmic lift combining cyclic phase and multiplicative scaling. These figures illustrate the common geometric structure of logarithmic coordinates as lifts from multiplicative domains into additive chart spaces.

Both encode the same topology. The difference is whether the lift exists as an analytic description or as an explicit constructive process.

Remark 3 (Symbolic Verification of the Discrete Lift)**.** *The discrete lift can be verified symbolically. For example, using SymPy one may define:*

```
import sympy as sp
q, r, s, k = sp.symbols('q r s k', integer=True)

# Eisenstein-like norm on the lattice
norm = q**2 + r**2 + s**2 - q*r - q*s - r*s

# Discrete logarithm in lifted state
lifted_log = sp.log(sp.sqrt(norm)) + sp.I * (sp.atan2(r, q) + 2*sp.pi*k)
```

Evaluating this expression for different values of k shows additive vertical layering without discontinuities, while projection to a fixed k reproduces seam-like discontinuities at branch boundaries.

Conclusion

Logarithms are geometric chart maps that lift multiplicative structure into extended coordinate spaces. This is visible in the classical Riemann surface of the complex logarithm, in everyday logarithmic measurement scales, and in orbital and spiral representations of multiplicative progression. The AOL formulation shows how the same lift can be realized constructively on a discrete substrate.

Glossary

Allen Orbital Lattice (AOL)
A discrete, two-dimensional lattice substrate used in Pattern Field Theory as a constructive foundation for geometry, state, and dimensional extension. In this paper, the AOL is used to realize logarithmic lifting as an explicit structural process rather than as an analytic covering space.

Branch Cut
A chosen discontinuity line in the domain of a multivalued complex function used to define a single-valued branch. In the logarithmic Riemann surface, branch cuts become seams along which sheets are glued. In the AOL formulation, branch cuts correspond to structural state transition boundaries.

Chart Map
A coordinate mapping that assigns numerical coordinates to elements of a domain. In this paper, logarithms are interpreted as chart maps from multiplicative (ratio) domains into additive coordinate spaces.

Covering Space
A topological space that "covers" another space such that each point in the base space has a neighborhood evenly covered by copies (sheets). The Riemann surface of the logarithm is a classical example.

Helicoid A spiral ramp-like surface generated by lifting angular rotation into a vertical dimension. The geometric shape naturally produced by the complex logarithmic lift.

Lift (Lifting)
The operation of extending a domain into a higher-dimensional space in order to represent otherwise multivalued or discontinuous structure as a continuous geometry.

Logarithmic Chart
A coordinate chart produced by applying a logarithm to a multiplicative domain, converting ratio composition into additive translation.

Logarithmic Dimensional Shift (LDS)
The principle that logarithms implement a genuine dimensional lift rather than merely a numerical reparameterization. In Pattern Field Theory, this lift is treated as a geometric and structural operation.

Multiplicative Domain
A space in which the natural composition law is multiplication or ratio rather than addition, such as positive real magnitudes, concentrations, or power ratios.

Riemann Surface
A connected topological surface used to represent a multivalued complex function as a single-valued function over a multi-sheeted domain.

Sheet Index
An integer label identifying which layer (or turn) of a multi-sheeted or lifted geometry a point lies on. In the classical theory this is implicit; in the AOL formulation it is an explicit state variable.

Structural State
A locally stored, explicitly represented variable in a discrete substrate that carries geometric or topological information, such as the sheet index in the AOL lift.

Winding Number
: An integer n counting how many times a closed path wraps around a singular core or origin. In logarithmic lifting, winding controls the additive sheet update $k \mapsto k + n$.

Dominion Index
: The integer-valued sheet state $k \in \mathbb{Z}$ used in the AOL lift to represent stacked layers of the logarithmic geometry as explicit structural state.

Representation Space
: A space used to describe or visualize a structure without necessarily being the physical or constructive substrate that generates it.

Structural Space
: A space whose elements and dimensions are generated by explicit constructive rules rather than by analytic extension or representation.

Helical Lift
: A lifted geometric representation in which cyclic motion in a base domain corresponds to translation along a helical or spiral surface in the extended space.

Orbital Lift
: A specific form of helical lift in which angular (orbital) phase and logarithmic scaling are combined to form a three-dimensional logarithmic helix. In Pattern Field Theory, this exists both as a continuous geometric model and as a discrete construction on the Allen Orbital Lattice via dominion stacking. This:

Decade A factor-of-ten change in a quantity. In logarithmic geometries, a decade often corresponds to one full turn or one fixed vertical increment.

Additive Coordinate Space
: A coordinate space in which composition is represented by addition rather than multiplication, typically obtained by applying a logarithmic chart to a ratio domain.

Discrete Lift
: A lifting operation implemented on a discrete substrate using explicit state updates rather than analytic continuation.

Document Timestamp and Provenance

This document is part of Pattern Field Theory (PFT) and the Allen Orbital Lattice (AOL). It extends the Logarithmic Dimensional Shift (LDS) framework and provides comparative geometric representations across classical and discrete substrates.