Pattern Field Theory
Finite Basins, Prime Indexing, and Structural Closure

Expanded Depth Series: Paper 4

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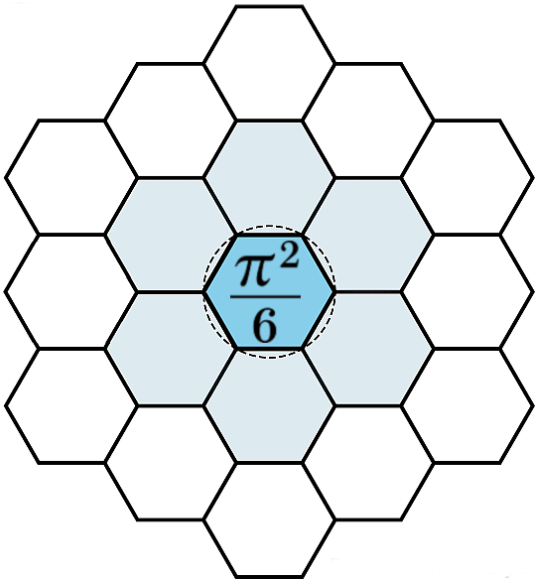

Abstract

This paper presents the formal construction of the Allen Orbital Lattice (AOL), the discrete structural substrate underlying Pattern Field Theory. The AOL is a prime-indexed hexagonal lattice defined over the Eisenstein integer ring $\mathbb{Z}[\omega] \setminus \{0\}$. Its geometry enforces finite basin capacity, structured recurrence, and constrained identity support without recourse to particles, background space, or probabilistic postulates. The purpose of this paper is to construct the lattice explicitly, define its adjacency and basin structure, and establish the invariants required for subsequent formulation of coherons, stability, and interaction.

1 Orientation and Scope

Pattern Field Theory asserts that physical identity is not fundamental to particles, trajectories, or fields defined over a background manifold. Instead, identity arises from structural closure within a discrete constraint geometry. The Allen Orbital Lattice (AOL) provides this geometry.

This paper is concerned exclusively with the construction and properties of the AOL itself. It establishes the lattice as a mathematical object and derives its structural consequences. No assumptions are made regarding particles, forces, measurement, probability, or dynamics. These topics are intentionally excluded and addressed in later papers of the Expanded Depth Series.

The reader is assumed to be familiar with the ontological premises of Pattern Field Theory as developed in Paper 1. No knowledge of coherons, Phase Alignment Lock, chemistry, or experimental phenomena is required or used here.

This document serves as the structural foundation upon which all subsequent formal developments depend.

2 Acronyms, Symbols, and Notation

- **PFT** — Pattern Field Theory
- **AOL** — Allen Orbital Lattice
- $\mathbb{Z}[\omega]$ — Eisenstein integer ring
- ω — primitive cube root of unity, $\omega = e^{2\pi i/3}$
- p — prime index
- r — shell radius
- k — basin index
- d — curvature depth

All symbols are used consistently throughout this paper and retain their meaning unless explicitly redefined.

3 Mathematical Domain: Eisenstein Integer Space

The Allen Orbital Lattice is defined over the Eisenstein integer ring $\mathbb{Z}[\omega]$, where

$$\omega = e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

Elements of $\mathbb{Z}[\omega]$ take the form

$$a + b\omega, \quad a, b \in \mathbb{Z}.$$

The Eisenstein integers form a two-dimensional lattice with hexagonal symmetry. Unlike square lattices derived from Gaussian integers, $\mathbb{Z}[\omega]$ minimizes angular anisotropy and admits uniform sixfold adjacency. This property is essential for enforcing consistent closure conditions across neighboring constraint paths.

The hexagonal structure induced by $\mathbb{Z}[\omega]$ is not a modeling preference but a structural necessity. Any lattice intended to support multi-path closure without privileged directions must satisfy this symmetry. Rectangular or cubic lattices introduce directional bias incompatible with the closure requirements of Pattern Field Theory.

The Allen Orbital Lattice excludes the origin element $0 \in \mathbb{Z}[\omega]$. This exclusion prevents degenerate self-alignment and enforces nontrivial adjacency relations across all admissible lattice nodes.

Prime indexing is imposed on lattice shells to enforce non-redundant structural separation between identities. Composite indices introduce factor degeneracy that collapses basin distinctness. The mathematical consequences of prime indexing are explored in subsequent sections.

4 Construction of the Allen Orbital Lattice

The Allen Orbital Lattice is constructed as a layered hexagonal lattice whose nodes are indexed by prime number and shell radius. Each node corresponds to a distinct admissible structural position within the lattice.

Shells are defined as concentric hexagonal layers centered on the excluded origin. The shell radius r is given by the minimal graph distance from the origin in $\mathbb{Z}[\omega]$ adjacency space. For each shell, nodes are assigned a prime index p according to an ordering rule that preserves adjacency consistency and basin separation.

Adjacency in the AOL is topological rather than metric. Two nodes are adjacent if their Eisenstein coordinates differ by a unit element. This adjacency relation defines permissible constraint coupling without invoking distance or embedding space.

Further properties of shells, basin boundaries, and adjacency invariants are developed in the sections that follow.

5 Shell Formation and Adjacency Rules

The Allen Orbital Lattice (AOL) is organized into discrete shells defined by topological distance within Eisenstein integer space. Shells are not geometric circles embedded in a background plane but graph-theoretic layers determined by adjacency relations.

5.1 Shell Definition

Definition 1 (Shell Radius)**.** *Let $v \in \mathbb{Z}[\omega] \setminus \{0\}$. The shell radius $r(v)$ is defined as the minimal number of adjacency steps required to reach v from the excluded origin under Eisenstein unit moves.*

Each shell S_r consists of all lattice nodes v such that $r(v) = r$. The first shell S_1 contains exactly six nodes corresponding to the unit elements of $\mathbb{Z}[\omega]$. Higher shells form finite hexagonal rings whose cardinality grows linearly with r.

Shells are finite by construction. No shell admits infinite extension, and no node belongs to more than one shell.

5.2 Adjacency Relations

Adjacency in the AOLis defined strictly through Eisenstein unit differences. Two nodes $v_1, v_2 \in \mathbb{Z}[\omega] \setminus \{0\}$ are adjacent if and only if

$$v_2 - v_1 \in \{\pm 1, \pm\omega, \pm\omega^2\}.$$

This adjacency relation induces a uniform sixfold neighborhood for all interior nodes. Boundary nodes at basin edges exhibit reduced adjacency but remain subject to the same rule set.

Adjacency is purely topological. No metric distance, angle, or embedding space is defined or required. The lattice supports constraint coupling solely through neighbor relations.

5.3 Consequences of Hexagonal Adjacency

The sixfold adjacency enforced by $\mathbb{Z}[\omega]$ is essential for supporting multi-path closure without directional bias. In lattices with fourfold or cubic adjacency, constraint loops preferentially align along axis directions, leading to degeneracy and unstable closure.

Hexagonal adjacency permits exactly three independent constraint axes, each with bidirectional pairing. This property will later be shown to be necessary for the existence of duplex curvature support and Phase Alignment Lock conditions.

5.4 Shell Connectivity and Closure

Within each shell S_r, adjacency relations form closed cycles. These cycles provide the minimal topological environment in which constraint paths may loop without intersecting the excluded origin.

Shell-to-shell adjacency is limited to nearest-neighbor shells S_{r-1} and S_{r+1}. No long-range adjacency exists. This restriction enforces locality of structural influence and prevents nonlocal identity coupling.

The shell structure therefore enforces both locality and closure simultaneously, a property that cannot be achieved in continuous or metric-based constructions.

5.5 Exclusion of the Origin

The explicit exclusion of the origin element is necessary to prevent trivial self-adjacency. Inclusion of the origin would permit degenerate zero-length constraint loops, collapsing shell distinction and eliminating basin boundaries.

By excluding the origin, all admissible closure paths necessarily traverse at least one shell, enforcing nontrivial structural identity.

This exclusion is structural, not symbolic, and applies universally across all depth levels of the AOL.

6 Prime Indexing and Non-Arbitrariness

The Allen Orbital Lattice (AOL) is not merely a hexagonal lattice; it is a *prime-indexed* lattice. Prime indexing is a structural requirement that prevents degeneracy, enforces identity separation, and guarantees finite basin capacity. This section establishes why prime numbers are necessary and why composite indexing fails.

6.1 Indexing Requirement

Each admissible lattice shell position is assigned an index $p \in \mathbb{P}$, where $\mathbb{P}$ denotes the set of prime numbers. The index does not label a particle, energy level, or occupation state. It labels a *structural capacity class* within the lattice.

Indexing by the natural numbers introduces unavoidable factor overlap. Composite indices permit multiple factorizations, allowing structurally distinct positions to collapse onto equivalent representations. Such collapse destroys identity separation and eliminates basin boundaries.

Prime indices, by contrast, admit no nontrivial factorization. Each index represents a unique irreducible structural unit.

6.2 Degeneracy Under Composite Indexing

Consider an indexing scheme based on the positive integers $\mathbb{N}$. For composite values $n = ab$, adjacency relations inherited from shell structure admit multiple equivalent decompositions. This induces degeneracy in closure conditions, allowing distinct constraint paths to satisfy identical indexing criteria.

Such degeneracy leads to uncontrolled identity overlap. Structural reuse becomes optional rather than enforced, and basin saturation loses meaning. In this regime, periodic recurrence is no longer inevitable but contingent, which is incompatible with observed structural regularity.

Prime indexing eliminates this failure mode entirely. No two primes share a nontrivial common factor, and no prime can be decomposed into smaller structural units.

6.3 Prime Separation and Identity Spacing

Prime indices enforce minimal separation between admissible identities. This separation is not spatial but structural: identities indexed by distinct primes cannot be continuously transformed into one another through adjacency-preserving operations.

Let $p_i, p_j \in \mathbb{P}$ with $p_i \neq p_j$. There exists no finite sequence of adjacency-preserving relabelings that maps p_i to p_j without violating shell or basin constraints. This property enforces discrete identity classes.

Prime separation therefore guarantees that identity reuse, when it occurs, must occur through exact recurrence rather than gradual deformation.

6.4 Non-Arbitrariness of Prime Assignment

The assignment of prime indices is not a choice parameter of the model. Any attempt to replace primes with alternative sequences (e.g., even numbers, Fibonacci numbers, or arbitrary labels) fails to preserve the irreducibility and separation properties required for stable lattice closure.

Primes are uniquely suited to this role because they are defined by the absence of internal structure. In the context of the AOL, this absence is precisely what prevents premature collapse of basin distinctions.

6.5 Structural Consequences

Prime indexing enforces the following properties:

- Finite basin capacity
- Mandatory identity reuse after saturation
- Discrete recurrence classes
- Immunity to factor-based degeneracy

These properties are structural consequences of the indexing scheme and do not depend on additional assumptions regarding dynamics, probability, or external constraints.

Prime indexing thus constitutes a foundational element of the Allen Orbital Lattice and cannot be removed without destroying its defining features.

7 Basin Definition

Prime indexing and shell structure together induce a finite partition of the Allen Orbital Lattice (AOL) into *basins*. A basin is not a spatial region in an embedding space; it is a bounded structural domain defined by admissible closure and adjacency constraints under a fixed depth regime.

7.1 Motivation

Shells provide layered organization and locality, while prime indexing enforces irreducible separation of structural classes. However, neither shells nor primes alone define where structural reuse must begin. Basin structure provides the boundary mechanism that distinguishes:

- a regime in which new identities may appear as distinct structural supports, and
- a regime in which identity reuse becomes mandatory due to finite capacity.

This definition is structural and does not depend on dynamics, measurement, or probabilistic interpretation.

7.2 Basin Boundary as a Closure Constraint

The AOLsupports identity only through admissible closure of constraint paths defined by local adjacency. A basin boundary is therefore defined as the maximal domain over which closure conditions remain internally consistent without requiring index reuse.

Intuitively: within a basin, the lattice admits a finite collection of distinct closure-supporting configurations; beyond that finite collection, any further attempt to introduce novelty forces collision with an existing structural class.

7.3 Formal Definition

Definition 2 (Basin)**.** *Fix a depth index d and the associated admissible rule set for structural support on the* AOL*. A basin $B_k(d)$ is a finite subset of lattice nodes such that:*

1. *(**Internal closure completeness**) For every admissible closure class under depth d and prime indexing, there exists at least one representative configuration supported entirely within $B_k(d)$.*
2. *(**Maximality**) $B_k(d)$ is maximal with respect to property (1): adding any node outside $B_k(d)$ does not increase the number of admissible distinct closure classes under depth d without inducing reuse of an existing prime-indexed structural class.*
3. *(**Boundary locality**) Any adjacency path that exits $B_k(d)$ crosses a boundary at which at least one closure constraint becomes unsatisfiable without index reuse.*

The basin index k labels the basin in the natural sequence induced by shell progression and prime allocation rules. The dependence on depth d is explicit: basin structure is defined relative to the admissible closure and support rules at a given internal resolution.

7.4 Basin Interior and Boundary

A practical distinction is useful:

- The **basin interior** consists of nodes whose admissible closure paths remain fully contained within the basin under the depth-d rule set.
- The **basin boundary** consists of nodes for which at least one admissible adjacency continuation would require either (i) leaving the basin or (ii) reusing an already-assigned prime-indexed structural class.

This interior/boundary distinction is purely structural. It is not defined by distance, radius, or metric properties.

7.5 Consequences

The basin definition immediately implies:

Proposition 1 (Finite Basin Capacity)**.** *For fixed depth d, each basin $B_k(d)$ is finite and admits only finitely many distinct closure-supporting structural classes before identity reuse is forced.*

The quantification of capacity and the mechanism of saturation are developed in the next section.

8 Basin Capacity and Saturation

The finiteness of a basin is not an empirical observation but a structural consequence of the Allen Orbital Lattice (AOL). For fixed depth d, adjacency rules, shell structure, and prime indexing together enforce a finite number of admissible closure-supporting configurations. This section formalizes basin capacity and defines saturation.

8.1 Capacity as a Structural Quantity

Basin capacity is defined as the maximum number of distinct structural identities that can be supported within a basin before prime-indexed reuse becomes mandatory. Capacity is not measured in spatial volume, energy levels, or occupancy counts. It is measured in admissible closure classes.

Let $\mathcal{C}_d$ denote the set of all closure-supporting configuration classes admissible under depth d. For a given basin $B_k(d)$, define

$$\mathcal{C}_d(B_k) \subset \mathcal{C}_d$$

as the subset of closure classes whose support lies entirely within $B_k(d)$.

Definition 3 (Basin Capacity)**.** *The capacity of basin $B_k(d)$ is*

$$\mathrm{Cap}(B_k(d)) := |\mathcal{C}_d(B_k)| \, .$$

By construction, $\mathrm{Cap}(B_k(d))$ is finite for all finite depth d.

8.2 Origin of Finiteness

The finiteness of basin capacity follows from three independent constraints:

1. **Finite shell cardinality**: Each shell contains finitely many lattice nodes.
2. **Local adjacency**: Closure paths may only traverse nearest neighbors, preventing unbounded combinatorial growth.
3. **Prime irreducibility**: Each admissible structural class requires a unique prime index, eliminating factor-based multiplicity.

Together, these constraints prevent unlimited introduction of distinct closure classes within a fixed basin.

8.3 Saturation

A basin is said to be *saturated* when all admissible closure classes under depth d have been instantiated within the basin.

Definition 4 (Basin Saturation)**.** *A basin $B_k(d)$ is saturated if for every closure class $C \in \mathcal{C}_d(B_k)$, there exists at least one structural identity in $B_k(d)$ realizing C.*

Once saturation is reached, no new distinct identity can be introduced without reusing an existing prime-indexed structural class. Novelty is structurally forbidden beyond this point.

8.4 Mandatory Identity Reuse

After saturation, any attempt to introduce additional structural identities forces reuse of an existing closure class. This reuse is exact: identities recur as structurally equivalent configurations rather than approximate deformations.

Proposition 2 (Mandatory Recurrence)**.** *For fixed depth d, any basin $B_k(d)$ admits identity recurrence once $\text{Cap}(B_k(d))$ is exhausted. No alternative configuration can be supported without violating adjacency, closure, or prime-indexing constraints.*

This recurrence is enforced by structure alone and does not depend on dynamical selection, energetic optimization, or probabilistic weighting.

8.5 Structural Implications

Basin saturation provides the structural origin of periodic recurrence observed in downstream domains. The inevitability of recurrence is not imposed by external rules but follows from finite capacity under fixed depth constraints.

The detailed mapping between basin saturation and observable recurrence patterns is developed in later papers of the Expanded Depth Series. Here it suffices to establish saturation as an unavoidable consequence of the AOLstructure itself.

9 Curvature and Depth

The Allen Orbital Lattice (AOL) admits an internal notion of curvature and depth that does not rely on embedding the lattice in a continuous space. Curvature in Pattern Field Theory is not defined through metric distortion of a manifold but through constraint load and adjacency tension within the lattice itself. Depth quantifies the admissible complexity of closure-supporting configurations under these constraints.

9.1 Curvature as Structural Load

In the AOL, curvature arises from the necessity to route multiple constraint paths through finite adjacency neighborhoods. When multiple closure paths compete for adjacency consistency, local constraint load increases. This load constitutes curvature in the Pattern Field Theory sense.

Curvature therefore measures the deviation from unconstrained adjacency, not the bending of a geometric surface. A region of high curvature is one in which closure paths must satisfy increasingly strict phase and adjacency conditions to remain admissible.

9.2 Depth Index

Depth is an internal resolution parameter that governs the allowable complexity of structural configurations. Increasing depth does not enlarge the lattice or introduce new nodes; it refines the admissible closure rules governing how paths may traverse existing adjacency structure.

Definition 5 (Depth Index)**.** *Let $d \in \mathbb{N}$ denote the depth index. Depth d specifies the maximal allowed constraint complexity for closure-supporting configurations on the* AOL*.*

At depth $d = 1$, only minimal closure paths are admissible. Higher depths permit nested, multi-loop, or compound closure structures, subject to the same adjacency and prime-indexing constraints.

9.3 Depth Increase Without External Expansion

Depth increase must not be interpreted as spatial expansion or dimensional augmentation. The underlying lattice remains fixed. What changes is the resolution at which closure conditions are evaluated.

Increasing depth therefore increases internal differentiation without invoking additional degrees of freedom. This distinction is central to Pattern Field Theory: complexity arises from internal refinement rather than external growth.

9.4 Interaction Between Curvature and Depth

Curvature and depth are coupled but distinct. Increased depth permits closure structures that distribute constraint load across multiple adjacency paths, thereby accommodating higher curvature without violating closure conditions.

Conversely, insufficient depth renders certain high-curvature configurations inadmissible. This interaction determines which identities can be supported at a given depth and which require a depth increase to become structurally viable.

9.5 Consequences for Basin Structure

Because basin capacity is defined relative to fixed depth, increasing depth modifies basin structure without altering lattice topology. New closure classes may become admissible, effectively increasing basin capacity while preserving prime-indexed separation and adjacency locality.

Depth increase thus provides a controlled mechanism for increasing internal resolution without collapsing structural invariants. This mechanism underlies the progressive refinement of identity support across basins.

10 Structural Invariants of the Allen Orbital Lattice

Despite increasing depth, basin progression, and identity refinement, the Allen Orbital Lattice (AOL) preserves a set of structural invariants. These invariants define what cannot change under any admissible transformation within Pattern Field Theory. They are not empirical regularities but consequences of lattice construction.

10.1 Adjacency Invariance

The adjacency relation defined by Eisenstein unit differences is invariant under all depth refinements and basin transitions. No operation within Pattern Field Theory alters which lattice nodes are neighbors.

This invariance ensures that locality is preserved at all resolutions. No depth increase permits nonlocal coupling, and no basin transition introduces long-range adjacency.

10.2 Prime Index Invariance

Prime indexing remains invariant across depth changes. Increasing depth refines closure rules but does not introduce new prime indices or modify existing ones.

As a result, identity separation remains discrete and irreducible. No process within the AOLallows a prime-indexed structural class to deform continuously into another.

10.3 Shell Topology Invariance

Shell structure is preserved under all admissible operations. While basin boundaries may shift as depth increases, the shell ordering induced by adjacency distance from the excluded origin remains fixed.

Shells neither merge nor fragment. This invariant guarantees a consistent notion of layered organization throughout the theory.

10.4 Basin Locality Invariance

Basin transitions occur only through saturation and depth refinement. No operation allows an identity to bypass basin boundaries or skip basin indices.

This invariant prevents premature identity mixing and ensures that recurrence arises only through mandatory reuse, not accidental overlap.

10.5 Depth Consistency Invariance

Depth refinement does not alter the underlying lattice. All nodes, shells, adjacency relations, and prime indices remain unchanged. Depth acts solely on the space of admissible closure configurations.

This invariant ensures that increasing resolution does not introduce new fundamental entities or degrees of freedom.

10.6 Summary of Invariants

The Allen Orbital Lattice preserves the following invariants under all admissible operations:

- Adjacency locality
- Prime irreducibility
- Shell topology
- Basin ordering
- Fixed lattice substrate

These invariants define the non-negotiable structural core of Pattern Field Theory. Any framework that violates them cannot support coherent identity under the constraints established in this paper.

11 Diagram Placement and Interpretation

Visual representations of the Allen Orbital Lattice (AOL) are used in this paper for structural clarification only. Diagrams do not introduce new assumptions, dynamics, or empirical claims.

They serve to illustrate shell structure, adjacency relations, and basin boundaries already defined mathematically.

11.1 Canonical AOL Diagram

The canonical diagram of the AOLshall be placed immediately after this section header. It must depict:

- The hexagonal lattice induced by $\mathbb{Z}[\omega]$
- Concentric shell structure centered on the excluded origin
- Local adjacency relations between neighboring nodes
- A representative basin boundary

The diagram should not include particle paths, force arrows, or metric distance labels. All visual elements must correspond directly to constructs defined in Sections 3–10.

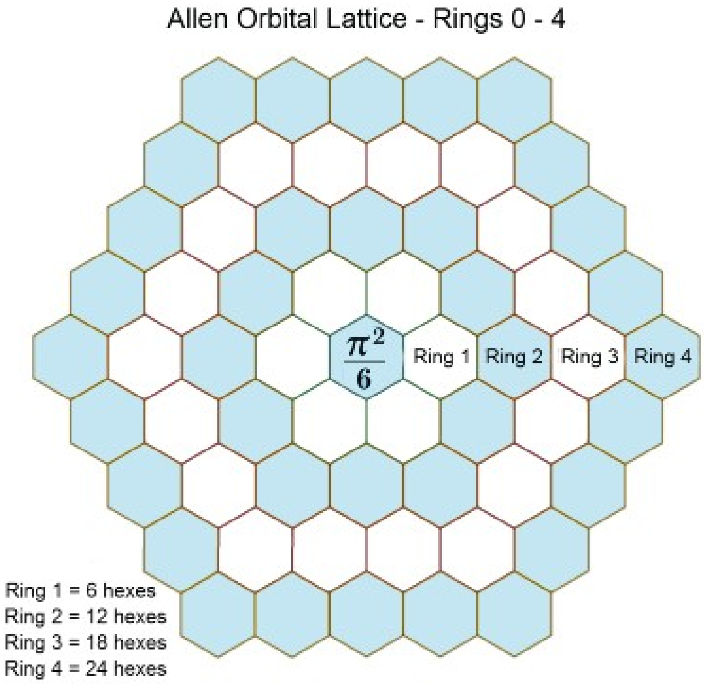

Figure 1: Canonical representation of the Allen Orbital Lattice. The hexagonal structure arises from Eisenstein integer adjacency. Concentric shells are defined by topological distance from the excluded origin. Basin boundaries delimit finite domains of admissible closure-supporting configurations under fixed depth.

11.2 Interpretation Guidelines

The diagram in Figure 1 must be interpreted structurally rather than geometrically:

- Node positions indicate adjacency, not spatial coordinates.

- Shell radii represent topological distance, not metric distance.
- Basin boundaries represent constraint limits, not physical barriers.

The diagram does not depict motion, propagation, or interaction. Such concepts are intentionally excluded from this paper and addressed in later works.

11.3 Relation to the PFT Logo

The Pattern Field Theory logo, placed on the title page, encodes the same hexagonal symmetry and prime-based structure illustrated here. Its purpose is symbolic and identificatory rather than explanatory.

The AOL diagram serves as the technical counterpart to the logo: where the logo signals structural symmetry, the diagram demonstrates its formal realization.

12 Glossary

Allen Orbital Lattice (AOL)
: The discrete hexagonal lattice defined over the Eisenstein integer ring $\mathbb{Z}[\omega] \setminus \{0\}$, serving as the structural substrate of Pattern Field Theory. The AOL enforces adjacency locality, finite basin capacity, and identity separation without reference to particles or background space.

Adjacency
: The topological neighbor relation between lattice nodes induced by Eisenstein unit differences. Adjacency defines permissible structural coupling and closure paths in the AOL.

Basin
: A finite structural domain of the AOL within which a fixed number of distinct closure-supporting identity classes can be realized under a given depth regime. Basins are defined by closure constraints rather than spatial boundaries.

Basin Capacity
: The maximum number of distinct closure-supporting structural classes admissible within a basin before identity reuse becomes mandatory.

Basin Saturation
: The structural condition in which all admissible closure classes within a basin have been realized, forcing exact identity recurrence for any further admissible configurations.

Closure
: The structural completion of a constraint path or set of paths within the AOL that satisfies adjacency, indexing, and depth constraints without violation. Closure is the criterion for identity support in Pattern Field Theory.

Curvature (PFT)
: A measure of constraint load arising from competing adjacency and closure requirements within the AOL. Curvature is defined structurally and does not correspond to geometric curvature of an embedding space.

Depth
: An internal resolution parameter governing the admissible complexity of closure-supporting configurations. Increasing depth refines allowable structural relations without modifying the underlying lattice.

Eisenstein Integers
: The ring $\mathbb{Z}[\omega]$ with $\omega = e^{2\pi i/3}$, whose algebraic structure induces hexagonal lattice symmetry.

Prime Indexing
: The assignment of prime numbers to structural identity classes to enforce irreducibility, separation, and finite basin capacity within the AOL.

Shell
: A set of lattice nodes at equal topological distance from the excluded origin, forming concentric hexagonal layers within the AOL.

Structural Invariant
: A property of the AOL that remains unchanged under depth refinement, basin progression, or admissible closure operations, such as adjacency locality or prime irreducibility.

13 Document Timestamp and Provenance

This document is issued as part of **Pattern Field Theory (PFT)** and belongs to the **Expanded Depth Series**. It provides a depth-expanded formalization of the **Allen Orbital Lattice (AOL)** as the structural substrate underlying all subsequent developments in the theory.

This paper does not revise or invalidate prior publications. Earlier documents remain valid expressions of Pattern Field Theory at their respective depth and resolution. The purpose of the Expanded Depth Series is to increase internal resolution, formal rigor, and instructional completeness while preserving the original axioms, direction, and discoveries of the framework.

All definitions, constructions, and invariants presented here are foundational and are treated as canonical for subsequent papers addressing coherons, stability, identity recurrence, chemistry, interaction, and experimental interpretation.