Pattern Field Theory
The Allen Orbital Lattice

Expanded Depth Series: Paper 2

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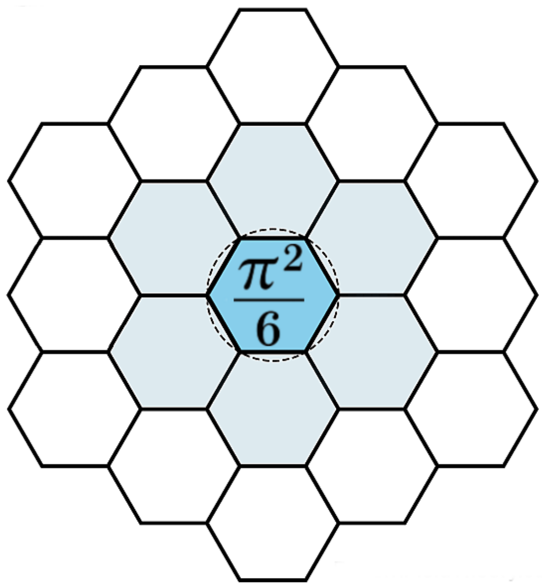

Abstract

This paper presents the full mathematical construction of the Allen Orbital Lattice (AOL), the structural substrate underlying Pattern Field Theory. The lattice is defined over $\mathbb{Z}[\omega] \setminus \{0\}$ and enforces hexagonal symmetry, prime-indexed recurrence, and finite basin capacity. The AOL is shown to be a constraint lattice rather than a spatial embedding, providing the structural conditions necessary for coherent identity, recurrence, and stability in later physical applications.

1 Orientation and Scope

This document is the second paper in the Expanded Depth Series of Pattern Field Theory (PFT). It assumes the ontological foundations established in Paper 1 and focuses exclusively on structural mathematics.

No physical interpretation is imposed beyond what is required to define the lattice itself. The Allen Orbital Lattice is presented as a mathematical object with intrinsic constraints, independent of measurement, particles, or fields.

The PFT pi-hex logo symbolically compresses the same hexagonal constraint geometry that is made explicit in Figure 1.

Why the Allen Orbital Lattice Is Not an Embedding Space

The Allen Orbital Lattice is not an embedding space and does not represent physical location, distance, or trajectory. Vertices of the lattice do not correspond to points in space, nor do edges represent paths of motion. Instead, the lattice defines a constraint architecture governing admissibility, recurrence, and compatibility relations between identities.

Adjacency in the AOL encodes structural compatibility rather than spatial proximity. Two vertices being adjacent indicates that transitions between their associated constraint states are admissible under the lattice rules, not that any displacement has occurred. No metric, background space, or continuous coordinate system is presupposed or introduced by the lattice construction.

This distinction is essential. Treating the AOL as geometry-in-space would incorrectly import assumptions from classical and relativistic frameworks that Pattern Field Theory explicitly rejects at the foundational level.

2 Expanded Acronyms

AOL Allen Orbital Lattice

PFT Pattern Field Theory

$\mathbb{Z}[\omega\ ] Eisensteinintegerringwith \omega = e^{2\pi i/3}$

$\mathbb{P}$ Set of prime numbers

3 Foundational Domain

Definition 1 (Eisenstein Domain)**.** *Let*

$$\mathbb{Z}[\omega] = \{a + b\omega \mid a, b \in \mathbb{Z},\ \omega^2 + \omega + 1 = 0\}.$$

The Allen Orbital Lattice is defined on $\mathbb{Z}[\omega] \setminus \{0\}$*.*

The exclusion of the origin is structural. The zero element corresponds to undifferentiated null and does not admit orbital identity.

Remark 1. *The use of* $\mathbb{Z}[\omega]$ *enforces hexagonal symmetry as a consequence of algebraic closure, not aesthetic choice.*

4 Hexagonal Necessity

Proposition 1. *Any lattice defined over $\mathbb{Z}[\omega]$ admits a unique nearest-neighbor structure forming a hexagonal tiling.*

Proof. The units of $\mathbb{Z}[\omega]$ generate six equidistant directions corresponding to the sixth roots of unity. Nearest-neighbor adjacency therefore forms a regular hexagonal graph. ☐

This hexagonal structure is intrinsic and cannot be replaced without violating ring closure.

5 Prime Indexing

Definition 2 (Prime Norm)**.** *For $z \in \mathbb{Z}[\omega]$, define the norm*

$$N(z) = z\bar{z}.$$

Prime elements are those whose norm is prime in $\mathbb{Z}$.

Proposition 2. *Prime-indexed vertices define irreducible recurrence anchors in the lattice.*

Prime indexing is structural, not symbolic: it enforces discrete recurrence points within a continuous constraint field.

Proposition 3 (Necessity of Prime-Based Addressing)**.** *Any indexing scheme permitting dense divisibility or continuous subdivision destroys finite basin admissibility and recurrence stability.*

Integer indexing admits uncontrolled factor reuse, while real-valued indexing collapses discrete identity boundaries through arbitrary subdivision. Both lead to degeneracy where structurally distinct configurations become indistinguishable under constraint evaluation.

Prime indexing enforces irreducibility. Each index acts as a minimal, non-decomposable structural address, preventing unintended collapse of identity classes while still permitting controlled recurrence under explicitly defined reuse conditions. This sparsity is not aesthetic; it is required to preserve basin finiteness and to support deterministic recurrence without ambiguity.

6 Orbital Shells and Basins

Definition 3 (Orbital Shell)**.** *An orbital shell $\mathcal{S}_k$ is the set*

$$\mathcal{S}_k = \{z \in \mathbb{Z}[\omega] \setminus \{0\} \mid N(z) = k\}.$$

Figure 1 makes explicit the discrete ring growth and finite shell geometry imposed by the Allen Orbital Lattice. The ring structure shown in the accompanying figure provides an intuitive visualization of finite basin growth via hexagonal adjacency. Formal basin shells in the Allen Orbital Lattice are defined by Eisenstein norm $N(z) = k$ and should not be interpreted as graph-distance rings; the diagram is illustrative of capacity saturation rather than a norm equivalence.

Definition 4 (Basin)**.** *A basin is a finite union of shells admitting stable recurrence under constraint propagation.*

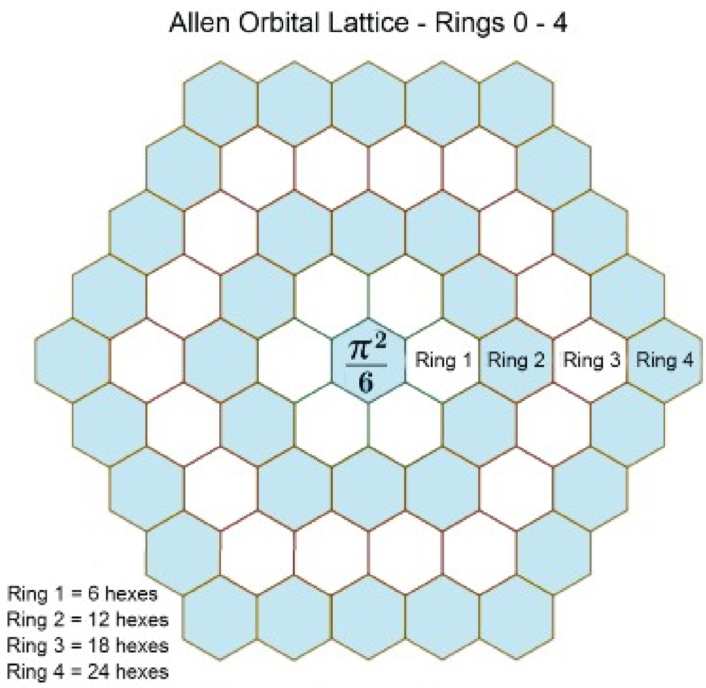

Figure 1: Allen Orbital Lattice growth illustration for rings 0–4. Each successive ring adds $6n$ hexagonal cells, establishing discrete shell growth and finite basin geometry. The central normalization term $\pi^2/6$ is used as the zeta-regularized normalization employed throughout Pattern Field Theory.

Proposition 4. *Every basin has finite capacity.*

Proof. The number of distinct shells below any fixed norm bound is finite. Recurrence constraints therefore saturate. □

This finiteness is the mathematical origin of periodicity.

It is critical to note that shell index does not correspond to distance, energy level, or physical scale. The shell structure encodes constraint saturation depth only. Progression through shells reflects increasing resolution and admissible identity complexity within the lattice, not motion through space or increase in magnitude of any physical quantity.

7 Curvature Depth

Definition 5 (Curvature Depth)**.** *Curvature depth is defined as the minimal number of shell transitions required to return a constraint path to equivalent orientation under lattice symmetry.*

Depth is an internal measure. No external metric is invoked.

Formally, curvature depth $d(\gamma)$ of a constraint loop γ is the minimal integer m such that

$$\gamma^m \sim \gamma$$

under lattice symmetry equivalence.

8 Lattice Invariants

The AOL admits invariants under:

- rotation by $2\pi/3$
- reflection across Eisenstein axes
- norm-preserving automorphisms

These invariants constrain all higher-level constructions.

9 Relation to Exceptional Structures

The Allen Orbital Lattice shares structural features with exceptional lattices, including E8, in its use of symmetry-enforced recurrence and constraint closure. However, the AOL is prime-indexed and basin-limited, distinguishing it from maximal symmetry lattices.

This distinction is essential and intentional.

10 Conclusion

The Allen Orbital Lattice provides the structural substrate of Pattern Field Theory. Its hexagonal symmetry, prime indexing, basin capacity, and curvature depth establish the necessary conditions for coherent identity and recurrence. Physical interpretation is deferred to subsequent papers.

Glossary

Prime Index Structural label assigned via the prime index map π

Allen Orbital Lattice Prime-indexed hexagonal constraint lattice defined over $\mathbb{Z}[\omega] \setminus \{0\}$.

Shell Set of lattice points with equal norm.

Basin Finite recurrence domain within the lattice.

Curvature Depth Internal measure of recurrence complexity.

Bibliography

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- Coxeter, H.S.M. *Regular Polytopes.* Dover, 1973.

Document Timestamp and Provenance

This document is part of Pattern Field Theory (PFT) and the Allen Orbital Lattice (AOL). It establishes the mathematical construction of the Allen Orbital Lattice (AOL), including prime indexing, shell structure, basin capacity, and lattice invariants, which are used by subsequent papers in the Expanded Depth Series.